Abstract

Recent work by Kaplan and Violante (2014) has shown that the composition of household portfolios, between low return assets that can be freely transacted and higher return assets that have some measure of illiquidity, can affect the efficacy of fiscal policy. In this paper I consider a two asset lifecycle model with transaction costs and show that allowing individuals to learn about their idiosyncratic skill (and therefore their future income distribution) à la Guvenen (2007) over time yields starkly different asset allocation and overall savings decisions by households as compared to a more restricted income process typically considered in the literature. Using parameters estimated from the same underlying income data as the standard process, I show that a model with learning generates more liquid saving over the entire lifetime, suppresses illiquid saving early on, and increases illiquid saving later in the lifecycle.
1 Introduction

The household portfolio allocation problem is not trivial. Household portfolios consist of a diverse array of assets, differing in many dimensions such as risk, maturity, and liquidity. Recent work by Kaplan and Violante (2014) has shown that the liquidity margin is essential for understanding the efficacy of fiscal policy. Individuals who are liquidity constrained desire, but are unable to finance, more consumption. These individuals will spend more out of a one off transfer payment than other agents. However, solely considering a household’s overall assets belies just how many households are constrained. Many households, though rich in wealth, lack liquidity as their net worth is tied up in difficult to adjust assets such as housing or retirement accounts. By decomposing household portfolios and considering an alternative measure of constrained households using only liquid assets, the authors are able to explain the substantial consumption response to tax rebates documented in Johnson et al. (2006). Thus, to understand the impact of policy, it is important to understand how and why individuals make decisions on the liquid-illiquid margin.

I consider a two asset model and show that the choice of exogenous income process is fundamental in household portfolio decisions. The functional form typically assumed in the literature is one in which income draws are a result of a very persistent shock, a transitory shock, and a deterministic trend component. Some papers, such as Kaplan and Violante (2014), allow for an idiosyncratic level component that is drawn at birth, fully known by the individual, and constant throughout the lifetime. This specification has the benefit of being highly tractable, as well as having some support in the data.

I consider an alternative specification, one in which individuals have both an idiosyncratic level and growth component, in addition to facing persistent and transitory shocks. These idiosyncratic components are drawn at birth and are constant throughout the lifetime, and can be thought of as an individual’s innate “skill.” This interpretation is both intuitively appealing and backed empirically; estimations using income panel data in Baker (1997) and Guvenen (2009) support this formulation. As in Guvenen (2007), I assume that skill is not directly observable by individuals and that they must learn about their earning potential over time. Individuals update their beliefs according to received income draws.

I incorporate this learning mechanism into an otherwise standard portfolio choice model. Individuals receive exogenous income draws throughout their working life and must save for retirement and/or precautionary reasons. Individuals have two accounts in which to save: a liquid account which can be adjusted freely and an illiquid account which bears a cost upon adjustment. To compensate for its lack of liquidity, the illiquid asset yields a higher return. I compare the portfolio decision implications of this model to one using the standard income setup, with no individual growth and the composition of all draws fully known by the individual.

In simulating the asset allocation decisions of individuals, I use parameters for each process that are estimated off of the same income data. Using a stylized calibration in which illiquid assets are

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1 See Cocco (2004), Scholz et al. (2006), Polkovnichenko (2007), Lynch and Tan (2011), among many others

fully irreversible until retirement, I show that despite the underlying income data being identical, the choice of income specification implies sharp differences in overall saving and portfolio allocation over the lifecycle. Overall saving is higher in the learning model throughout the lifetime. Further, individuals in the learning model shift their illiquid savings to later in life and begin decumulating liquid savings earlier in life as compared to their benchmark counterparts. Average liquid asset holdings are higher throughout the entire working lifecycle in the learning model, as well. I argue that the benchmark model generates too little liquid holdings as compared to illiquid holdings. Briefly comparing these findings to data from 2013, I show that the learning model’s overall ratio of average illiquid assets over the lifecycle to average liquid assets over the lifecycle is more in line with the ratio in the data. Moreover, if one considers the median individual’s ratio of illiquid to liquid assets throughout their lifetime, the benchmark ratio is always higher than in the data. The learning model does better at matching this ratio at the beginning of the lifecycle, though the model does underpredict this ratio in the middle of the median individual’s life.

These results are driven by the way the learning model affects lifetime income uncertainty and the mean forecasts of agents. Lifetime uncertainty is higher when individuals must learn their skill level, particularly early in the life. Further, due to the negative population covariance between the level and growth components of income, prosperous individuals in the learning model are highly pessimistic early in their life. Later in life, individuals who receive high income eventually become more optimistic than their counterparts in the standard model. These two factors contribute to the different savings patterns in the two models.

Thus, the model with heterogeneous skill and learning has potentially different policy implications than standard models. The increase in liquid holdings in the learning model implies that individuals will be less liquidity constrained, potentially diminishing any stimulative fiscal policy. If one is calibrating the model to match the number of constrained households, as is done in Kaplan and Violante (2014), then other parameters would have to adjust, such as the coefficient of risk aversion, which would then in turn affect the response to policy. The message of this paper extends to other applications as well. Housing is an obvious case, since it dominates household balance sheets. Iacoviello and Pavan (2013) create a general equilibrium lifecycle model incorporating housing. They consider the effects of changes in income risk and down payments on various macroeconomic variables and model the effect of countercyclical financial conditions on the housing market and household balance sheets. Fisher and Gervais (2011) use a life cycle model to understand why home ownership has fallen among young individuals, and conclude that an increase in earnings risk and decrease in marriage accounts for the trend. Yang (2009) uses borrowing constraints and transaction costs to model household consumption over the lifecycle. These papers incorporate the standard model of income risk. If one considers the illiquid asset to consist of just housing, then this paper implies that there may be additional insights to be gleaned from the exercises considered in the literature.

This paper also fits into the literature seeking to understand how incorporating an income process with learning into a lifecycle model affects household decision making. Guvenen (2007) shows that in
a one asset environment, allowing households to learn their skill level over time has success in matching
the inequality of consumption over the lifecycle. Guvenen and Smith (2014) use indirect inference
to estimate a one asset model of consumption and savings where individuals learn their idiosyncratic
growth rate. They find that the persistent component mean reverts much faster than commonly believed
and that individuals know around half of their skill level immediately upon entering the work force. As
in this paper, Chang et al. (2016) consider a two asset model. However, their margin of interest is
the one between “safe” assets and riskier assets with a stochastic return. They combine learning with
unemployment risk and occupational switching to produce a model that can reproduce the documented
finding that the risky share of an individual’s portfolio increases with age. Though similar in spirit to
this paper, their margin has different policy implications compared to the liquid-illiquid margin. Finally,
to the extent that this paper provides some evidence for the learning process being an adequate model
for another aspect of household behavior, this paper adds to the debate on the appropriate way to think
about income.

The structure of this paper is as follows. Section 2 considers household portfolio data, and elaborates
on the distinction between liquid and illiquid assets. Section 3 gives details on the model, solution
strategy, and calibration while Section 4 examines the mechanisms of the model in greater detail. Section
5 discusses the main results of the paper and Section 6 concludes.

2 Data

To get a sense of household portfolio composition, I consider data from the Survey of Consumer Finances
(SCF). The SCF is a triennial survey conducted by the Federal Reserve since 1983. The survey contains
very limited data on income, demographics, and expenditures, though as the name might suggest it is
noted for its rich and exhaustive consumer financial data. I partition this data into groups, liquid and
illiquid, only slightly deviating from the methodology of Kaplan et al. (2014). Liquid assets consist of
checking and savings accounts, money market funds, call accounts, directly held mutual funds or stocks,
liquid bonds (such as Treasury bills or corporate bonds), miscellaneous financial assets such as gold or
silver, and cash. Cash is a rare asset that is not recorded by the SCF. Thus, I further follow Kaplan
et al. (2014) by imputing a household’s cash holdings using the ratio of average cash to checking, savings,
money market, and call accounts found in Foster et al. (2013). For the sake of brevity, for the
remainder of this analysis, I group checking, savings and cash together, money market and call accounts
together, and combine directly held mutual funds and stocks as “stocks”, all liquid bonds into “bonds”,
and miscellaneous financial assets into “other liquid assets”.

Illiquid assets consist of residential and nonresidential properties, retirement accounts (such as a 401K
or IRA) and annuities, certificates of deposit (CDs), the total cash value of any life insurance policies,
trusts, savings bonds (which cannot be transferred), any stake in a non-actively managed business, and
miscellaneous non-financial assets such as jewelry, expensive paintings, baseball cards, etc. I exclude
vehicles, as I consider these short-term consumption items, rather than long-term investments. Further, I
exclude the value of actively managed businesses of entrepreneurs. I group together secondary residential and nonresidential properties as “non-primary property”, and consider primary housing as a separate entity. I group retirement accounts and annuities together as “retirement assets”, and combine CDs, life insurance, trusts, savings bonds and stake in non-actively managed businesses into “illiquid financial” assets. Miscellaneous illiquid assets I denote “illiquid nonfinancial” assets.

Because the focus of this paper is on learning about ability while working, I put everything in terms of experience, rather than age. “Experience” is not provided by the SCF. Therefore, I create an experience variable by subtracting the highest grade completed (plus six, the standard age to begin schooling) from the age of the head of household. For individuals who have less than ten years of schooling, I subtract sixteen from the head, as this is the minimum age an individual can begin working. These surveys are fairly vague when it comes to postgraduate work. There is no way of telling from the data how long an individual has spent in graduate school, or when they graduated; one can only tell whether the individual attended graduate school at some point in their life. I consider any graduate school attendance as one more year of schooling, and assume that the individual attended immediately after completing their undergraduate work.

Table 1: Average Share of Liquid Assets

<table>
<thead>
<tr>
<th>Experience</th>
<th>Total Value</th>
<th>Cash/Checking/Saving</th>
<th>MM/Call</th>
<th>Stocks</th>
<th>Bonds</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>90,852.96</td>
<td>70.11</td>
<td>8.18</td>
<td>14.77</td>
<td>.55</td>
<td>3.46</td>
</tr>
<tr>
<td>≤ 4</td>
<td>6,527.043</td>
<td>79.23</td>
<td>4.49</td>
<td>8.4</td>
<td>0</td>
<td>4.76</td>
</tr>
<tr>
<td>5 - 9</td>
<td>10,902.03</td>
<td>75.68</td>
<td>5.57</td>
<td>10.33</td>
<td>0</td>
<td>5.5</td>
</tr>
<tr>
<td>10 - 14</td>
<td>20,219.41</td>
<td>76.06</td>
<td>5.58</td>
<td>10.82</td>
<td>0</td>
<td>5.02</td>
</tr>
<tr>
<td>15 - 19</td>
<td>41,210.54</td>
<td>72.18</td>
<td>7.61</td>
<td>12.94</td>
<td>0</td>
<td>4.38</td>
</tr>
<tr>
<td>20 - 24</td>
<td>60,990.56</td>
<td>70.61</td>
<td>7.06</td>
<td>15.41</td>
<td>.19</td>
<td>3.83</td>
</tr>
<tr>
<td>25 - 29</td>
<td>93,701.5</td>
<td>69.15</td>
<td>7.98</td>
<td>15.67</td>
<td>.52</td>
<td>3.85</td>
</tr>
<tr>
<td>30 - 34</td>
<td>128,618.1</td>
<td>67.84</td>
<td>9.04</td>
<td>16.72</td>
<td>.41</td>
<td>3.29</td>
</tr>
<tr>
<td>35 - 39</td>
<td>157,850.5</td>
<td>66.97</td>
<td>8.74</td>
<td>18.34</td>
<td>.74</td>
<td>2.66</td>
</tr>
<tr>
<td>40 - 44</td>
<td>184,044.4</td>
<td>64.16</td>
<td>9.97</td>
<td>18.28</td>
<td>1.13</td>
<td>4.18</td>
</tr>
<tr>
<td>45 +</td>
<td>143,585</td>
<td>67.34</td>
<td>10.5</td>
<td>15.85</td>
<td>1.65</td>
<td>1.79</td>
</tr>
</tbody>
</table>

Table 2: Average Share of Illiquid Assets

<table>
<thead>
<tr>
<th>Experience</th>
<th>Total Value</th>
<th>Residence</th>
<th>Other Property</th>
<th>Retirement</th>
<th>Other Fin</th>
<th>Other Nonfin</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>260,677.9</td>
<td>51.88</td>
<td>7.6</td>
<td>24.42</td>
<td>13.06</td>
<td>.75</td>
</tr>
<tr>
<td>≤ 4</td>
<td>14,797.75</td>
<td>23.71</td>
<td>3.49</td>
<td>28.77</td>
<td>33.79</td>
<td>9</td>
</tr>
<tr>
<td>5 - 9</td>
<td>42,920.9</td>
<td>37.40</td>
<td>5.18</td>
<td>33.19</td>
<td>18.82</td>
<td>2.43</td>
</tr>
<tr>
<td>10 - 14</td>
<td>76,493.99</td>
<td>45.95</td>
<td>5.03</td>
<td>30.68</td>
<td>14.47</td>
<td>1.12</td>
</tr>
<tr>
<td>15 - 19</td>
<td>140,625.5</td>
<td>49.00</td>
<td>6.91</td>
<td>28.26</td>
<td>12.64</td>
<td>.97</td>
</tr>
<tr>
<td>20 - 24</td>
<td>206,609.5</td>
<td>50.71</td>
<td>6.32</td>
<td>29.31</td>
<td>9.73</td>
<td>.75</td>
</tr>
<tr>
<td>25 - 29</td>
<td>292,519.3</td>
<td>51.58</td>
<td>8.34</td>
<td>27.72</td>
<td>9.24</td>
<td>.64</td>
</tr>
<tr>
<td>30 - 34</td>
<td>379,477.1</td>
<td>51.99</td>
<td>9.07</td>
<td>27.61</td>
<td>8.17</td>
<td>.62</td>
</tr>
<tr>
<td>35 - 39</td>
<td>449,188.4</td>
<td>52.19</td>
<td>9.44</td>
<td>26.27</td>
<td>8.51</td>
<td>.66</td>
</tr>
<tr>
<td>40 - 44</td>
<td>493,504.5</td>
<td>53.52</td>
<td>9.92</td>
<td>23.07</td>
<td>9.99</td>
<td>.60</td>
</tr>
<tr>
<td>45 +</td>
<td>352,829.2</td>
<td>62.66</td>
<td>7.94</td>
<td>11.16</td>
<td>15.41</td>
<td>.46</td>
</tr>
</tbody>
</table>
I construct Tables 1 - 4 to get at the relative importance of each asset in the typical household’s portfolio. In Tables 1 and 2 I calculate the average share of each asset class as a percentage of total liquid and illiquid assets for different levels of experience, as well as for the entire sample. Splitting the sample up in this way gives some insight into whether some assets gain importance at certain points in the lifetime, which could potentially be lost by just looking at the sample overall. In order to attenuate the impact of outliers, I Winsorize each at the 5th and 95th percentile for each group. In the tables I also include the average holdings of each type of asset, to give some sense of how large each asset class is in the average household’s portfolio.

The implications of these tables are clear. The average household balance sheet is dominated by a few assets. Households hold an average of 65 - 70% of their liquid assets in easily accessible checking or savings accounts, or directly as currency, though the share of this asset declines over the lifecycle. They hold an additional 5 - 10% in money market accounts or call accounts, which bear little risk and are nearly as easy to access as checking accounts. Put together, these near-riskless, highly liquid assets make up nearly 80% of the average household’s liquid assets. Bonds play nearly no role in the average household’s balance sheet at any point throughout the lifetime; the average household holds just 1/2% of their liquid assets in non-savings bonds. Miscellaneous financial assets, such as precious metals, are nearly as unimportant to the average household, accounting for only 3.5% of their liquid assets. Directly held stocks and mutual funds increase as a percentage of liquid assets throughout most of the lifecycle, with a slight dip for individuals who have been out of school for more than forty-five years. Directly held stocks and mutual funds are the second most important of these assets, accounting for around 15% of the average households liquid portfolio. This amount is still dwarfed, however, by near-cash assets.

Illiquid assets are similarly dominated by only a few types. Primary residences account for over half of all illiquid holdings; secondary residences and non-residential structures account for another ≈ 8%. Further, retirement accounts and annuities comprise on average 25% of a household’s illiquid assets. Miscellaneous nonfinancial assets, such as valuable personal collections, are nearly a non-factor for the average household, at less than 1%. Non-retirement accounts, such as CDs or trusts, are the largest asset on average for individuals at the very beginning of their careers. However, for the vast majority of their working life these assets are small compared to retirement accounts and properties. Overall, the average household holds 13% of their illiquid wealth in these assets.

It may seem curious to consider the mean, rather than the median here. This is because, at least for the entire sample, the median holding of all assets other than housing, retirement accounts and checking accounts is zero, or nearly zero. Nevertheless, it may be the case that certain assets, while not important for the median household, may comprise a large percentage of the assets of a minor but substantial subset of households. Therefore, in Tables 3 and 4 I compute the percentage of households for whom each asset comprises a certain percentage of each asset type.

Tables 3 and 4 do not contradict the implications of Tables 1 and 2 regarding the relative importance of each asset to households. On the liquid side, it is clear that near-cash assets dominate the portfolios of most households. 97% of households have at least some of their assets as near-cash, and a majority of
Table 3: % of Households With Minimum Share of Each Liquid Asset

<table>
<thead>
<tr>
<th>Minimum Share</th>
<th>Cash/Checking/Saving</th>
<th>MM/Call</th>
<th>Stocks</th>
<th>Bonds</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>96.82</td>
<td>21.27</td>
<td>25.9</td>
<td>2.9</td>
<td>10.77</td>
</tr>
<tr>
<td>5</td>
<td>90.99</td>
<td>18.71</td>
<td>24.88</td>
<td>2.62</td>
<td>9.62</td>
</tr>
<tr>
<td>10</td>
<td>84.24</td>
<td>16.47</td>
<td>23.75</td>
<td>2.26</td>
<td>8.74</td>
</tr>
<tr>
<td>25</td>
<td>76.01</td>
<td>12.44</td>
<td>20.57</td>
<td>1.59</td>
<td>6.88</td>
</tr>
<tr>
<td>50</td>
<td>68.30</td>
<td>8.10</td>
<td>15.39</td>
<td>.86</td>
<td>4.83</td>
</tr>
<tr>
<td>75</td>
<td>62.15</td>
<td>4.78</td>
<td>9.66</td>
<td>.39</td>
<td>3.37</td>
</tr>
<tr>
<td>90</td>
<td>58.68</td>
<td>2.55</td>
<td>4.88</td>
<td>.15</td>
<td>2.22</td>
</tr>
<tr>
<td>99</td>
<td>56.45</td>
<td>0</td>
<td>.56</td>
<td>.02</td>
<td>1.22</td>
</tr>
</tbody>
</table>

Table 4: % of Households With Minimum Share of Each Illiquid Asset

<table>
<thead>
<tr>
<th>Minimum Share</th>
<th>Residence</th>
<th>Other Property</th>
<th>Retirement</th>
<th>Other Fin</th>
<th>Other Nonfin</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>79.37</td>
<td>21.7</td>
<td>57.73</td>
<td>46.39</td>
<td>8.72</td>
</tr>
<tr>
<td>5</td>
<td>78.56</td>
<td>19.65</td>
<td>52.00</td>
<td>33.81</td>
<td>6.31</td>
</tr>
<tr>
<td>10</td>
<td>76.72</td>
<td>17.56</td>
<td>46.86</td>
<td>26.36</td>
<td>4.61</td>
</tr>
<tr>
<td>25</td>
<td>68.78</td>
<td>12.3</td>
<td>35.24</td>
<td>16.34</td>
<td>2.65</td>
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<tr>
<td>50</td>
<td>52.13</td>
<td>6.40</td>
<td>20.84</td>
<td>9.98</td>
<td>1.63</td>
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<td>75</td>
<td>36.24</td>
<td>3.03</td>
<td>11.43</td>
<td>7.10</td>
<td>1.15</td>
</tr>
<tr>
<td>90</td>
<td>26.29</td>
<td>1.78</td>
<td>8.11</td>
<td>6.023</td>
<td>.94</td>
</tr>
<tr>
<td>99</td>
<td>17.83</td>
<td>1.12</td>
<td>6.72</td>
<td>5.46</td>
<td>.79</td>
</tr>
</tbody>
</table>

households hold nearly all of their liquid assets (>99%) as near-cash. Money market and call accounts are less popular; only 21% of households have any funds in these accounts and less than a tenth of households keep the majority of their liquid assets in these accounts. Few households have any exposure to bonds; only 2.9% of households have even 1% of their liquid portfolio as bonds. Miscellaneous liquid assets are fairly small; a tenth of household own some of these assets, but they only comprise a majority of liquid assets for less than 5% of households. For a subset of the population, stocks play a large role in their liquid portfolio. For 10% of households, stocks are over 75% of their liquid holdings. However, stocks are unimportant for the majority of households, as nearly 75% of household keep less than 1% of their liquid portfolio in equities.

Illiquid portfolios are dominated by primary residences. Nearly 80% of households have some share of their illiquid holdings in their primary residence. The majority of households hold the majority of their illiquid holdings in their home, and 18% of households hold nearly their entire illiquid portfolio in their home. A fifth of households have some equity in other properties, and these properties account for over a quarter of illiquid wealth for a tenth of households. Retirement accounts, too, are important. A majority of households have assets earmarked for retirement, and a fifth of households hold the majority of their illiquid holdings in these accounts. A tenth of households have more than 75% of their illiquid portfolio in retirement accounts. Non-retirement financial instruments are important to households as well. 47% of households have some exposure to these assets and 17% of households have at least a quarter of their illiquid assets in these instruments. Miscellaneous nonfinancial assets are not crucial to understanding household balance sheets. Less than a tenth of households have any of these assets,
In Figure 1, I graph aggregate measures of portfolio composition. I consider data from the 2013 SCF. Because the illiquid asset data are for the most part reported as net asset data, I subtract from liquid assets any outstanding debts, such as credit card debt, educational loans, or installment loans. I exclude vehicle debt since I have excluded vehicle worth from each category. Thus, in Figure 1 it is best to think about these measures as assets net of any corresponding debts.

In Figure 1a, I plot the average holdings of liquid and illiquid assets over the lifecycle. The average holding of illiquid assets is always higher than that of liquid. Further, both series are single peaked at around 45 years of experience, which is slightly after retirement for individuals who begin working after college. Average liquid assets are actually negative at the beginning of working life, though this measure becomes positive fairly quickly. Illiquid assets are decumulated rapidly in retirement, while average liquid assets decline at a much slower pace.

In Figure 1b, I consider the ratio of illiquid to liquid assets, to get a sense of the distribution of portfolio composition. I plot the ratio for 25th percentile, the 75th percentile, and the median individual at each level of experience. Households tend to being life with little illiquid assets; thus the ratio for each percentile is near zero at the beginning of their career. The 75th percentile steadily rises over the lifecycle, almost linearly. By retirement the 75th percentile has 25 times illiquid net worth as compared to liquid net worth. These ratios are not nearly as large for lower percentiles. The median individual increases his relative illiquid holdings over the lifecycle as well, though by sixty years of experience this ratio only rises to 6. The 25th percentile’s ratio actually becomes negative during mid-life, implying negative holdings of one type of asset. Nevertheless, after 60 years of experience the ratio is positive. However, it is less than one, implying that liquid holdings are higher than illiquid holdings.
3 Model

Income Process and Learning

The most general version of the income process can be thought of as a combination of three factors. The first is deterministic demographic considerations, such as number of children or experience, and is denoted $D_{i,e}$, where $e$ is the individual’s level of experience. This component is fully predictable, completely observable, and affects everyone in the same way. The second factor is an idiosyncratic component, such as inherent “skill”, denoted $X_{i,e}$. This factor is unobservable to the econometrician, and perhaps even to the individual themselves. Note that this individual component can potentially change as the individual gains experience. The last component is random chance, i.e. “shocks”, and is denoted by $\Xi_{i,e}$. To summarize, one can write that individual $i$ in time $t$ has income $Y_{i,e}$:

$$Y_{i,t} = D_{i,e}X_{i,e}\Xi_{i,e}$$

Taking logs, one gets the following:

$$y_{i,e} = d_{i,e} + x_{i,e} + \xi_{i,e}$$

I assume that demographic effects are linear in their arguments, that is that $d_{i,e} = \Phi_{t} \text{demo}_{i,t}$. Further, I subtract this from log income to get $\tilde{y}_{i,e}$, which is the portion of an individual’s income draw that is due to factors unobservable by the econometrician. As mentioned, these factors can either be due to individual-specific factors or mere randomness.

I follow Guvenen (2007) and the subsequent literature in assuming that the individual component is a linear function of experience:

$$x_{i,e} = \alpha^{i} + \beta^{i}e$$

Notice that neither the intercept term, $\alpha^{i}$, nor the growth term, $\beta^{i}$, have experience subscripts. These objects are parameters for each individual that are drawn at birth and are constant throughout the individual’s lifetime. These parameters are assumed to be jointly normal and may be correlated:

$$\left(\alpha^{i}, \beta^{i}\right) \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{\alpha}^{2} & \sigma_{\alpha\beta} \\ \sigma_{\alpha\beta} & \sigma_{\beta}^{2} \end{bmatrix}\right)$$

The shock process, $\xi$, can be thought of as an ARMA($p,q$). For clarity, I separate it into two components, a moving average process $\varepsilon$ and an autoregressive process $z$. Though $\varepsilon$ could be modelled as any length of MA($q$), for tractability I assume that $\varepsilon$ is white noise. Further, in the literature $z$ is often taken to be an AR(1), which is the assumption that I will make in this paper. To summarize:

$$z_{i,e} = \rho z_{i,e-1} + \eta_{i,e}$$
\[ \eta \sim N(0, \sigma_\eta^2) \]
\[ \varepsilon \sim N(0, \sigma_\varepsilon^2) \]

Thus, one can rewrite the portion of income that is due to factors unobservable to an outside observer as the following:

\[ \tilde{y}_{i,e} = \alpha^i + \beta^i \varepsilon + z_{i,e} + \varepsilon_{i,e} \]

Notice that this process nests the more straightforward case found in the majority of the literature. This typical case restricts \( \beta^i = 0 \), and often holds \( \alpha^i = 0 \) as well. In the case that \( \alpha^i \) is drawn from some distribution, individuals are fully aware of this value. In either case, any individual uncertainty in \( \tilde{y} \) is driven completely by the unpredictable persistent and transitory shocks. For the remainder of this paper, I consider the “benchmark” model to be the one that allows for heterogeneity in individual intercepts. Allowing for this generality affects the distribution of initial income, but not the uncertainty the individual faces per se.\(^3\)

The individual has knowledge about his earning potential to the extent that he knows \( \alpha^i, \beta^i \), and the current value of the persistent component \( z_{i,e} \) (and therefore implicitly the size of the transitory shock, \( \varepsilon_{i,e} \)). However his understanding of these components may be imperfect. Therefore, the individual uses his income draws to gradually learn about his earning potential over time. The individual understands all of the parameters of the income process, such as the underlying variances, and of course observes his income. He never observes \( \{\alpha^i, \beta^i, z_{i,e}, \varepsilon_{i,e}\} \), but rather updates his expectations and uncertainty surrounding these objects through observable income draws.

The individual may have some prior information about his idiosyncratic components. I set priors up in a similar way to Guvenen (2007). One can partition the \( \alpha^i \) and \( \beta^i \) into components the individual has full information on, \( \alpha^i_k \) and \( \beta^i_k \), and components for which the individual has no information, \( \alpha^i_u \) and \( \beta^i_u \), such that:

\[ \alpha^i = \alpha^i_k + \alpha^i_u \]
\[ \beta^i = \beta^i_k + \beta^i_u \]

Let the within-parameter components be mean zero and independently drawn, that is, \( \alpha^i_k \perp \alpha^i_u \) and \( \beta^i_k \perp \beta^i_u \), and let the variances of \( \alpha^i_k, \alpha^i_u, \beta^i_k, \) and \( \beta^i_u \) be \( \sigma_{\alpha,k}^2, \sigma_{\alpha,u}^2, \sigma_{\beta,k}^2, \) and \( \sigma_{\beta,u}^2 \), respectively. Then the variances of \( \alpha^i \) and \( \beta^i \) are

\[ \sigma_\alpha^2 = \sigma_{\alpha,k}^2 + \sigma_{\alpha,u}^2 \]
\[ \sigma_\beta^2 = \sigma_{\beta,k}^2 + \sigma_{\beta,u}^2 \]

\(^3\)Note however that any estimation that allows for \( \alpha^i \neq 0 \) can potentially return different estimates of the variance of the persistent and transitory components than a more restrictive estimation. The estimates of these parameters may affect implied uncertainty.
I denote \( \lambda_\alpha \) and \( \lambda_\beta \) as the fraction of variance of the intercept and growth component, respectively, that is immediately resolved for the individual. In other words, before the individual has even received his first income draw, his prior uncertainty about \( q \in \{ \alpha^i \beta^i \} \) is \( \sigma^2_{q,u} = (1 - \lambda)\sigma^2_q \). It follows that his prior for the covariance between the components is \( \sqrt{(1 - \lambda_\alpha)} \sqrt{(1 - \lambda_\beta)} \sigma_{\alpha\beta} \).

The individual updates his beliefs via Kalman Filtering. The agent is concerned with the components \( [\alpha^i \beta^i z_{i,e}]' \) that govern his income. Based on the draws up to and including his current level of experience, the individual forms expectations of this vector, \( \hat{S}_{e|e} \). Further, his uncertainty about these parameters is denoted by \( P_{e|e} \), the variance-covariance matrix of the components given his experience level. The individuals prior variance-covariance matrix is:

\[
P_{0|0} = \begin{bmatrix}
(1 - \lambda_\alpha)\sigma^2_\alpha & \sqrt{(1 - \lambda_\alpha)} \sqrt{(1 - \lambda_\beta)} \sigma_{\alpha\beta} & 0 \\
\sqrt{(1 - \lambda_\alpha)} \sqrt{(1 - \lambda_\beta)} \sigma_{\alpha\beta} & (1 - \lambda_\beta)\sigma^2_\beta & 0 \\
0 & 0 & \sigma^2_\eta
\end{bmatrix}
\]

Prior expectations are based on the initially resolved, known portion of the component partition: \( \hat{S}_{0|0} = [\alpha^k_i \beta^k_i 0]' \). Note that the individual’s prior belief about the persistent component will always be zero as this is a random shock that is completely independent of his idiosyncratic and demographic characteristics. The evolution of his expectations and the variance-covariance matrix thereafter is standard and therefore the details are left to the Appendix.

**Household Problem**

I consider a relatively standard two asset lifecycle model. Individuals work and receive exogenously generated income, \( Y_t \), for \( T \) years, and then retire for an additional \( T_{Ret} \) years, during which they receive no income. Individuals supply labor inelastically during their working years and receive utility only from the consumption good, \( c_t \). I assume that utility is time-separable. That is, in each period \( s \), agents make decisions to maximize the following:

\[
U(c_t, c_{t+1}, ...) = \sum_{t=s}^{T+R_t} \delta^{t-s} u(c_t)
\]

where \( \delta \) is the household’s discount factor.

Households can save either through a liquid or illiquid asset, denoted by \( m_{t+1} \) and \( b_{t+1} \), respectively. Households are free to accumulate or decumulate the liquid asset without incurring any cost. Adjustment of the illiquid asset is costly, however, and carries a proportional transaction cost. A proportional transaction cost was chosen for this environment to mimic costs faced in adjusting a household’s housing stock and retirement accounts (such as 401ks or IRAs), the two most important illiquid assets on most household’s balance sheets. Housing transaction costs can include attorney fees, real estate agent commissions, and transfer taxes, all of which scale with the value of the home transacted. Retirement accounts face a 10% withdrawal penalty\(^4\). Of course, generalizing all of the transaction costs associated

\(^4\)There are a few exceptions to the withdrawal penalty, such as when the withdrawal is used to purchase a first home or
with these assets in this way is not perfect: one can easily argue for the existence of fixed costs for housing purchases, and individuals can freely deposit into retirement accounts. However, for the illustrative purposes of this stylized model, proportional costs will suffice.

Transaction costs must be reinvested in the illiquid asset or those returns will incur a cost, as is the case for any investments made within a retirement account. This also matches how returns are accessed in the case of housing. Households must sell their home, and pay the corresponding transaction costs, to permanently withdraw any equity from capital gains. Total transactions costs, the fraction \( \gamma \) of the change in the illiquid account, can therefore be written the following way:

\[
tc_t = \gamma |(1 + r^b) b_t - b_{t+1}|
\]

As briefly alluded to previously, the transaction costs associated with expansion and reduction of these assets may differ. To allow for this generality, \( \gamma \in \{ \gamma, \hat{\gamma} \} \), where \( \gamma \) is applied to withdrawals \( (b_{t+1} < (1 + r^b)b_t) \) and \( \hat{\gamma} \) is charged for deposits \( (b_{t+1} > (1 + r^b)b_t) \). All individuals begin their life with zero assets; that is \( b_0 = 0 = m_0 \). Further, individuals cannot borrow against any asset at any point in time: \( b_{t+1} \geq 0; m_{t+1} \geq 0 \forall t \).

Liquid assets yield a net return of \( r^m \). To compensate for their relative lack of liquidity, illiquid assets carry a higher return than that of the liquid asset: \( r^b > r^m \). These returns are time-invariant and do not carry any risk.

When making consumption and savings choices, households care not only about current conditions but also the distribution of future income realizations. Since the current income realization \( Y_t \) is a combination of different factors, which all have very different implications for future income, this object alone is not very informative about future prosperity. Thus, the household’s value function must take into account the full vector of individual beliefs about the underlying structure of his current income draw, \( \hat{S}_{t|t} \), as this value influences his expectations about the future, \( E^S \). Further, since beliefs evolve with additional information, the individual must internalize how his beliefs with change from an income draw in the next period when understanding his expected continuation value. So, the household’s value function can be written as follows:

\[
V_t(Y_t, \hat{S}_{t|t}, m_t, b_t) = \max_{c_t, b_t, m_t} u(c_t) + \delta E_t^S[V_{t+1}(Y_{t+1}, \hat{S}_{t+1|t+1}, m_{t+1}, b_{t+1})]
\]

In retirement, the problem is somewhat simplified. Since there are no transaction costs in retirement, the individual no longer has use for the liquid asset and therefore dumps all of his savings into the illiquid asset in his final working year. Further, in retirement the individual no longer receives any income and therefore faces no uncertainty. The dynamic problem becomes:

\[
V_t(b_t) = \max_{c_t, b_t} u(c_t) + V_{t+1}(b_{t+1})
\]

pay for certain medical expenses, among other things. I abstract away from these exceptions for the rest of the paper.
In other words, this is a cake eating problem in which the individual must determine the optimal amount of his initial retirement assets to consume in each retirement period. This can be solved for analytically and necessarily satisfies the following set of equations:

\[(1 + r^b)b_t = c_t + b_{t+1}\]

\[c_{T+t} = c_{T+1}[\delta(1 + r^b)]^{\frac{t+1}{T+1}}, \forall t\]

\[\sum_{t=1}^{R_t} \frac{1}{1 + r^b} c_{T+t} = b_{T+1}\]

where the first set of equations are the Euler equations and the second equation is the budget constraint for the entire retirement period.

**Solving the Model**

The model is solved via backwards induction. For a given level of assets, accumulated during the working periods, the total utility derived from the retirement periods can be solved for analytically and summarized as one value, \(V_{Ret}(b_{T+1})\). Note that \(m_{T+1} = 0\) as there are no transaction costs in retirement, but the illiquid asset still yields a higher return. Further, due to the lack of uncertainty the only relevant state variable is the level of illiquid assets, \(b_{T+1}\).

Numerical methods are necessary to retrieve the decision rules throughout the individual’s working life. Individuals make their portfolio selections, \(\{m_{t+1}, b_{t+1}\}\), off of a \(M x B\) grid, where \(M = 100\) and \(B = 100\). The grid is not evenly spaced. To make sure that individuals do not have to adjust their illiquid holdings unless they desire to, each dimension of the grid is spaced such that \((1 + r^b)q_i = q_{i+1}\), where \(q \in \{m, b\}\) and \(q_i\) and \(q_{i+1}\) are sequential grid points. This allows individuals to choose their previous holdings (plus accrued interest) as their current asset holdings, without having to incur a cost or commit more of their assets to a less liquid position. In every period, the choice set for each asset is the same, so differences between grid spacing does not bias individual decision making towards any one asset.

The value function of the agent depends not only on their current income, \(y_t\), but also their beliefs about how much of that income can be attributed to \(\alpha^i\), \(\beta^i\), and \(z^i\). Thus, there are four additional state variables that one must keep track of while computing the decision rules. Notice that for a long enough time horizon \(y_t\) is unbounded, even if income shocks are truncated. Thus, it is of some use to replace \(y_t\) in the individual’s problem with the forecast error, \(\epsilon_t\).

Notice that, conditional on his expectations about the state today, from the individual’s perspective his next-period income follows the following distribution:

\[y_{t+1} | s_{t+1} \sim N(H|s_{t+1} + P|H_{t+1} + R)\]

where \(H, P,\) and \(R\) are defined in the Appendix. Defining \(\epsilon_{t+1} = y_{t+1} - s^\alpha_{t+1|t} - s^\beta_{t+1|t}(t + 1) - s^z_{t+1|t}\), it is
clear that $\epsilon_{t+1}$ is normally distributed with mean zero and a variance of $H'_{t+1} P_{t+1} H_{t+1} + R$. Notice that the variance here does not depend on individual income draws, as $P_{t+1}$ evolves solely according to the variances of the components of the income process. Thus, the variance for $\epsilon_t$ can be solved separately from backward induction. For a given period’s variance, I create a 5-spaced grid for $\epsilon_t$, adapting the method introduced in Tauchen (1986) for the case where $\rho = 1$. Extreme values are given by three standard deviations above or below the mean, zero.

The value function is a function of the individual’s beliefs in that period. Thus, when computing the expected value of the following period’s value, one must account for how beliefs will change between periods. Next period beliefs are a function of current beliefs and the next period income draw, restated above as $\epsilon_{t+1}$. Thus, for each of the gridpoints of $\epsilon$, I calculate the values of $S_{t+1|t+1}$ given $S_{t|t}$:

$$s_{t+1|t+1} = s_{t|t} + K_{t,q} \epsilon_{t+1}$$

where $K_{t,q}$ is the cell of the time-t Kalman Gain $K_t$ corresponding to component q and $S_{t|t} = \exp(s_{t|t})$.

I use a $5 \times 5 \times 5 \times T$ grid to store the values at various beliefs. Since beliefs evolve on a continuum, typically the individual’s next period beliefs do not fall on the grid. I use trilinear interpolation in these cases. The endpoint values of the grid change with the time period, $t$, and reflect the most extreme values the grid can take given the possible income draws of $\epsilon$ in the current period. Thus, there is no need for extrapolation.

The computational strategy of the benchmark process is fairly straightforward and discussed in the Appendix.

**Calibration**

The household has constant relative risk-aversion (CRRA) preferences over consumption: $u(c_t) = c_t^{1-\sigma} / (1-\sigma)$, where $\sigma$ is chosen to be 2. I consider each period to be a year. I set total working years, $T = 43$. This aligns with the longevity of an individual who enters the labor force upon graduation from college at age 22 and retires at the age of 65. I set $R_t = 15$, which implies that individuals die at age 80, which is only slightly higher than the unconditional life expectancy in the US of 79. Since I consider time to be annual, I set the discount factor to $\delta = .96$. I set the interest rate for the liquid asset such that it completely offsets the discount factor; that is $r^n = \frac{1}{\delta} - 1$. To compensate for the increased risk in the illiquid asset, I set a positive wedge between the interest rates such that $r^b = r^n + .02$. For the remainder of the paper I consider the “no-prior” case; that is, $\lambda_\alpha = 0 = \lambda_\beta$, and assume that mean income in each period, $D_t$ is normalized to 1.

My parameterization of the adjustment cost is highly stylized, which is done primarily for expositional

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5In 2015, according to the World Bank
purposes. I set $\bar{\gamma} = 0$; in other words I completely waive costs for households adjusting their illiquid assets upward. This is not necessarily an unrealistic assumption for many types of illiquid assets. For example, retirement accounts levy a penalty only if funds are withdrawn before a certain age, and do not penalize deposits. The main deviation from reality is my assumption of $\gamma = 1$, or complete irreversibility, which is perhaps only applicable to a highly specific subset of assets, such as irrevocable trusts. While this choice has the effect of increasing the risk of holding the illiquid asset, it also makes the choice between liquid and illiquid assets quite stark. Because of their complete irreversibility under this parameterization, illiquid assets are accumulated solely for use in retirement periods as they are unavailable to smooth any severity of shocks during working periods. Thus, in this setting liquid assets are the only way to save for use in later working periods. Thus this parameterization makes it very clear what amount of savings the individual plans to use for retirement (illiquid savings) and what amount of savings is held either for precautionary purposes or to use later in the lifecycle, perhaps if the individual expects income to fall (liquid savings).

Table 5: Income Process Calibration

<table>
<thead>
<tr>
<th>Model</th>
<th>$\rho$</th>
<th>$\sigma_\alpha^2$</th>
<th>$\sigma_\beta^2$</th>
<th>$\sigma_{\alpha\beta}$</th>
<th>$\sigma_\eta^2$</th>
<th>$\sigma_z^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learning</td>
<td>.805</td>
<td>.023</td>
<td>.00049</td>
<td>-.0024</td>
<td>.025</td>
<td>.032</td>
</tr>
<tr>
<td>Benchmark</td>
<td>.979</td>
<td>.031</td>
<td>-</td>
<td>-</td>
<td>.0099</td>
<td>.047</td>
</tr>
</tbody>
</table>

I calibrate the income process to parameters estimated in Guvenen (2009). First isolating and removing the deterministic trend, Guvenen then uses GMM to match moments from the income residuals. In this way he is able to estimate each model in turn using income data alone. Both processes are estimated from the same male earnings data, taken from the Panel Survey of Income Dynamics (PSID). See that paper for more details on sample selection and method.

There are significant differences between the parameterizations of the two processes. Dispersion of the intercept is much higher in the benchmark case. However, note that this high level of dispersion does not contribute in any way to uncertainty in the benchmark model, as individuals receive this draw at birth and are fully aware of its value. This value merely affects the initial variance of income draws and each individual’s idiosyncratic mean income. The transitory component does impact uncertainty in both models, and is higher in the benchmark. Further, the benchmark has a higher persistence parameter. The difference is substantial; only $\approx 14\%$ of a persistent shock remains after 10 periods in the learning process as opposed to a little over 82% in the benchmark process. However, the variance of the persistent shock is higher in the learning model, so how the individual feels about this trade-off depends on how heavily the individual discounts future periods.

Crucially, individuals in the learning model have an idiosyncratic growth component whereas those in the benchmark model do not. The small variance of this component belies its importance in determining income, especially later in life. An individual who has a growth parameter equal to one standard deviation above the mean has $\beta_i = .0221$, or a level of income $\approx 2\%$ higher in the first period of working.

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6 Though the tax-exemption deposits receive ends at a certain threshold. Since I abstract from this feature of retirement accounts entirely, I ignore this distinction.
and $\approx 25\%$ higher after the 10th period of working, when compared to an individual who has a mean growth of 0. Finally, notice that the population covariance between the intercept and growth component is actually negative. The correlation between these two components is considerable: -$0.7059$.

## 4 Intuition

The household’s asset decisions depend crucially on the type of exogenous income process they are facing. This is because a model with learning changes both the structure of uncertainty over the lifetime as well as the individual’s forecast of their mean income. I consider both of these mechanisms in turn.

To get a grasp on how uncertainty resolves itself over the lifetime, I consider the present discounted value of the square root of the individual’s mean squared error of log income in both the learning model and under the benchmark. This is because the mean squared error of log income can be written as follows:

$$MSE_{t+s|t} = E_t[(\hat{y}_{t+s} - y_{t+s}|t)^2]$$

$$= E_t[(\hat{y}_{t+s} - \hat{y}_{t+s}|t)^2]$$

$$= Var_t(\hat{y}_{t+s} - \hat{y}_{t+s}|t)$$

$$\approx Var_t\left(\frac{\hat{Y}_{t+s} - \hat{Y}_{t+s}|t}{\hat{Y}_{t+s}}\right)$$

where $\hat{Y} = \frac{Y}{D}$ and the final approximation follows from the convenient properties of logs, as long as forecast error is small. Thus, one can think of the mean squared error of the forecast in period $(t+s)$ from the vantage point of period $t$ as the variance of the percentage forecast error. The square root of this object gives the percentage deviation from mean income of a one standard deviation increase in percentage forecast error.

In order to get a sense of lifetime uncertainty, I then multiply the percentage deviation by mean income, which is normalized to $D_t = 1$ in each period. This gives the dollar amount of a one standard deviation increase in forecast error. For each period $t$, I then discount each of these deviations by the household’s discount factor, and sum them for each future period $(t+s)$:

$$\sum_{s=1}^{T} \delta^s D_{t+s}\sqrt{MSE_{t+s|t}}$$

This gives the total (discounted) change in income if the household’s forecast error was one standard deviation above the mean in every future period.

In Figure 2a I plot this amount for each period for both the learning case as well as the benchmark, using the values from the calibration section. Figure 2b is the same information, just normalized so that initial uncertainty is equal to one in both cases. It is clear that over the entire lifecycle uncertainty is greater is the learning model, with the greatest disparity being towards the beginning of the lifecycle.
Further, from the normalized graph it is clear that initially uncertainty is resolved much quicker in the benchmark case, while the pace of uncertainty resolution is higher towards mid-life in the learning case. After period 25 or so both models imply nearly equivalent uncertainty resolution for the rest of the lifecycle.

For the benchmark case, uncertainty resolves itself almost linearly. This results entirely from the individual’s lifespan shortening over time. As they gain experience, there are simply fewer periods left for the persistent shock to accumulate; thus their uncertainty falls as the variance of the transitory component is constant over the lifecycle. Since the periods that drop off are highly discounted at the beginning of life and are only slightly discounted latter on, there is some concavity, albeit slight.

The graph depicting the learning model similarly features declining uncertainty through the attenuation of the accumulation of persistent shocks. The shape of the uncertainty profile in this model is of course also driven by the individual’s lack of information about his own skill parameters. Early in life, the individual knows little about his true level of skill. Though his lack of knowledge about his intercept contributes to initial uncertainty, this high level of uncertainty is mostly due to his lack of information on his growth potential. Income in mid-life and later years is dominated by the growth potential of the individual, and therefore even small changes in this draw can yield large changes in total lifetime income.

Further, since this component does not contribute much to income draws early in life, it takes a while before the individual can tighten the distribution of this component. Thus, uncertainty resolves itself very slowly early in life under the learning model, and even increases slightly as individuals age closer to (and therefore discount less) the periods where the growth rate matters faster than they learn about it. Eventually income draws become more informative, and the individual is able to get a better sense of what his growth component truly is. This is reflected in each of the above graphs by a quick decline in lifetime uncertainty in mid-life. By the second half of the lifecycle, the individual has a good idea of
what his growth potential is, and additional observations yield small gains. At this point uncertainty is resolved primarily by declining possible accumulation of the persistent component, though meagre gains from additional learning about the skill components contributes as well. The speed of uncertainty resolution is similar in both models during this period, as well.

An individual’s conditional mean forecasts of future income are also different over the lifetime between the two models. To illustrate, I consider demeaned log income, $\tilde{y}$. In the benchmark case, forecasts are fairly straightforward. Individual know how much of their income is due to each component. Thus, they know the value of their intercept and consider it fixed; they allow for mean reversion of the persistent component; and they assume that the transitory component will completely dissipate. Thus, given $\tilde{y}_t = \alpha + z_t + \varepsilon_t$, conditional expectations for period $(t+s)$ are:

$$E_t[\tilde{y}_{t+s}] = \alpha + \rho^s z_t$$

In the learning model individuals of course do not know the value of their intercept, growth rate, persistent component, or transitory component, but merely have beliefs about these components based on their priors and information in the form of income draws. Thus, the actual composition of the current income draw does not matter for forecasts; only the composition of the individual’s beliefs. The individual assumes that their intercept and growth terms will remain constant, and use the later component to extrapolate growth in future income. Further, the individual assumes the transitory component will immediately disappear and the persistent component will deteriorate geometrically with time. Thus, one can write the conditional forecast as so:

$$E_t[\tilde{y}_{t+s}] = s_{t+s}^\alpha + (t+s)s_{t+s}^\beta + \rho^s s_{t+s}^\gamma$$

Notice that, while the forecasts of the benchmark model evolve the same way irrespective of starting period $t$, this is not the case in the learning model. This is because in the learning model forecasts are based on current beliefs, and current beliefs can vary depending on the timing within the lifecycle. To illustrate this point I compare the forecasts for individuals whose current income draw is one standard deviation above the mean at various points in the lifecycle. For ease of exposition I assume that in the benchmark case $\alpha = 0$. If $\alpha$ were allowed to vary the forecast would still mean revert at the same rate; it would merely mean revert to $\alpha$ rather than 0. For our purposes here this distinction is unimportant. Further, because contemporaneous beliefs depend on the entire sequence of income draws, there is no one “true” forecast for an individual in the learning model. Therefore I simulate income draws for 100,000 individuals, and average the expectation path of the 100 individuals immediately above and 100 individuals immediate below the one standard deviation mark. The graphs are normalized so that the initial period’s income is equal to 1.

The most striking result here is that, although benchmark forecasts are invariant to the period, forecasts in the learning model depend heavily on the individual’s experience level. As mentioned before, the benchmark forecasts revert to the mean at rate $\rho$ in all periods. For the learning model, the
Figure 3: Income Expectations

(a) Mean Forecast at Period 1

(b) Mean Forecast at Period 5

(c) Mean Forecast at Period 15

(d) Mean Forecast at Period 30
entire expectation path changes shape as the expectation level increases. In period 1, mean forecasts are highly pessimistic. Individuals predict their income will not only fall immediately, but will continue to decline over their entire lifetime. In fact, individuals believe that their income will eventually fall to less than mean income.\footnote{Note that this exercise deals with de-meaned income, and that the mean income profile is increasing over most of the lifecycle. Thus, these individuals are not necessarily predicting that their actual income will fall over their lifetime; only that their income relative to the mean will decline.} Forecasts in period 5 decline rapidly as well, though they merely asymptote to mean income. By period 15, forecasts attain more of a U-shape, declining for the first 7 or so periods and then actually increasing for the rest of the lifecycle. The trough is at 70% of initial income. Forecasts in period 30 have a shallower U-shape, declining to only 95% of initial income.

The forecasts are a result of which components the individual attributes his positive income draw to. The individual knows the population distributions of each of the parameters, and understands the potential of each component to affect his income throughout his life. Due to the low variance of the growth component, the individual understands that this component will most likely not account for much of his early income draws, but will dominate his income path later in life. The opposite is true of the intercept component, which will be important early in life but have very little effect later on, relative to the growth component. Further, he understands that the intercept and growth components negatively covary, and that a positive intercept will tend to be accompanied by a negative growth rate.

Thus, an individual who receives a very high income draw in the first period knows that it is likely not due to the growth component. Therefore, he attributes some of this draw to having a positive intercept, some of the draw to receiving a positive persistent shock, and the rest to receiving a positive transitory shock. Further, he believes that the growth component is actually negative, due to the negative covariance in the population between the intercept and growth components. Thus, the individual’s forecast is a combination of mean reversion and an expectation of negative growth, relative to the mean. This leads to low expected income throughout the lifecycle, despite receiving a high income draw.

By period 15, individual forecasts change dramatically as compared to those earlier in life. At this point the growth component does account for a non-negligible portion of income draws, and individuals receiving high income can assume that their growth component is positive. They still believe that some portion of their income is due to a positive persistent shock; this accounts for the decline in expected income over the first few subsequent periods as the shock diminishes faster than their income increases due to the growth component. Eventually, around 7 periods later the growth component wins out and individuals begin forecasting increasing income. After 20 periods the individual expects income to be higher than it is in the initial period, and the individual anticipates even higher income thereafter.

Forecasts in period 30 tell a similar story. However, consider that, due to the widening of the income distribution over the lifecycle, an income draw one standard deviation above the mean in period 30 is much higher than it is in period 15. Further, on average individuals who receive a high draw in period 30 have received more high draws throughout their lifetime than an individual who receives a high draw in period 15. Both of these factors lead individuals who receive this draw to attribute proportionally more of their income to a high growth component and less to the persistent component. Thus, the U-shape is...
Individuals at different income levels within each period have different forecasts, as well. In Figure 4 I plot out the forecasts of individuals at one standard deviation and one half standard deviation above the mean in period 1 and again for period 15. One can tell with even a cursory look at these graphs that individuals with income draws closer to the mean tend to have less “extreme” forecasts; that is, their forecasts deviate less from their current income than individuals who have current income values further away from the mean. This is because, with smaller income draws, individual’s have lower beliefs about the growth and persistent components, which drive the forecast dynamics.

For example, in period 1 a lower draw leads individuals to believe both the persistent and intercept components are lower. Though they still believe the growth component is negative, because the intercept component is closer to the mean value of zero their expectation for growth is closer to zero as well. Thus, individuals who receive less income expect less mean reversion (since the base is smaller) and less negative growth in the future. So their expectations, while still pessimistic, are less drastic than individuals who receive higher income in this period. In period 15, individuals who receive an income draw only one half standard deviation above the mean have lower beliefs of the persistent and growth component. This leads them to expect less mean reversion in the near term and lower growth over the rest of the lifecycle. This imples a shallower U-shape than that of the individuals who receive a higher draw.

Though I have omitted them from the graphs, expectations are symmetric around the mean. That is, individuals who receive an income draw one standard deviation below the mean have the opposite forecast from those who receive a draw one standard deviation above the mean. This implies that individuals with a poor draw in period 1 are actually fairly optimistic and believe that eventually they will receive income above the mean. Individuals who continue to receive negative draws throughout
their lifetime become more and more pessimistic, however.

5 Results

I simulate the lifecycle paths of income, consumption, and assets for 1000 individuals in both models using decision rules found via backward induction. There may be some concern that, because these two processes yield different income distributions over the lifecycle (see Appendix), the results discussed here may be more of a function of the distribution rather than from the mechanisms emphasized above. Thus, I use the same income draws, generated from the benchmark process, for each simulation. The individuals in the learning simulation still act as if they are receiving draws from the learning process and update their beliefs accordingly. Though not reported here, the results discussed in this section are robust to using different income draws for each model generated by their respective parameter estimates. Each individual has an unconditional expected income of 1 in the first period.

5.1 Comparison of the Two Models

Figure 5 shows the population average of various measures of household savings over the lifecycle. All individuals begin their life with zero assets, and choose to save for two reasons. The first is precautionary, to buffer consumption from unpredictable shocks. Due to the curvature of the utility function, expectations over the marginal utility of uncertain draws is greater than the marginal utility of the expectation of the draws. Thus, the more uncertainty they face, the more individuals suppress consumption today in favor of greater saving. The second savings motive is to make up for future expected shortfalls in income. This includes the lengthy retirement period, in which individuals receive zero income (and therefore infinitely negative utility if they have no savings; i.e. death) as well any anticipation of declining earnings during the individual’s working periods.

In Figure 5a, I graph the total assets- liquid and illiquid- that the average household has accumulated at each experience level. In both models average savings increase monotonically until retirement, and then decline thereafter as individuals draw down their savings to fund retirement consumption. Due to the absence of any bequest motive (or possibility of an early, unanticipated death), all individuals end their lifetime with zero net assets. Average savings in the learning model are higher throughout the lifecycle as compared to the benchmark case. Much of this can be attributed to the higher level of uncertainty individuals face throughout their lifetime, increasing the precautionary motive and therefore the average level of savings. There are two things to note here. First, though income uncertainty is gradually resolved throughout the lifetime in both models, theoretically decreasing the amount of desired savings, average savings never falls during working periods. This is because any precautionary savings that are freed up due to less uncertainty must then be spread over the rest of the individual’s working life and retirement, so little of it translates into consumption and most remains held as savings. Further, while the disparity in uncertainty between the two models falls as individuals age, at every point there is more uncertainty in the model with learning. Thus, at every experience level the precautionary motive
Average savings are higher in the learning model partially due to expectations, at least in the early periods. High income individuals who learn are more pessimistic about their future income path early in life than their benchmark counterparts, and therefore save more of their income to smooth consumption. Similarly, low income individuals are more optimistic than if they had the benchmark process, and therefore would like to dissave and consume more. Note, however, that low income individuals will in general have low levels of savings built up, particularly this early in their life. This, combined with the inability of individuals to borrow, introduces an asymmetry. While high income individuals do not face constraints on adjusting their savings, low income individuals do, and cannot dissave as much as they would like. Therefore, in both models average savings increase due to expectations of mean reversion. However, these expectations are exacerbated in the learning model, and therefore the increase in average savings is higher than in the benchmark model.

Notice that the peculiarities of the transaction cost restrict the aforementioned savings motives to specific assets. Because illiquid savings cannot be withdrawn before retirement, any savings accumulated for use during the individual’s work life must be deposited into the liquid account. This means that any savings done for precautionary reasons will be placed in the liquid account, as income is deterministic in retirement. Further, any saving undertaken in anticipation of a decline in earnings during the individual’s working periods must be stored in the liquid account as well as that is the only way they can be accessed when needed. Similarly, since the illiquid asset can be freely accessed in retirement and earns a higher return, any funds saved specifically for retirement will be placed into the illiquid account.

In Subsection 5b I decompose total savings up into its constituent parts for each model. The general, qualitative pattern of saving is similar in each model. Average illiquid savings are strictly increasing until retirement, after which time they are slowly drawn down. Average liquid savings begin to rise immediately and are hump shaped over the working years, peaking at around twenty or thirty years of experience, depending
on the model. In the final period of work individuals dump all of their liquid assets into the illiquid account and maintain a zero liquid balance thereafter. Quantitatively, the two models exhibit stark differences in average portfolio allocation. Individuals who learn about their income over time hold on average less savings in their illiquid account until they have around 30 years of experience, despite saving more overall. Afterwards, this relationship flips for the rest of the lifecycle with learning individuals holding more illiquid assets on average. At the time of retirement, average illiquid holdings in the learning model are around 2 units higher than in the benchmark case. This is fairly substantial, given that median income is normalized to 1 unit. Throughout all working periods, the learning model produces a higher level of liquid savings. This difference between the models can be large: at its peak, average liquid savings in the learning model are more than four times the peak in the benchmark model. Further, average liquid holdings reach their maximum point at different experience levels. Individuals who learn begin decumulating liquid assets on average in their early twenties. For the benchmark agents, this does not occur until their late twenties.

This discrepancy in portfolio composition is driven by the way the two income processes influence risk and expectations. The higher precautionary motive not only causes individuals in the learning model to save more, but also to direct more of their savings into the liquid fund, which they can access in case of poor earnings in the future. Similarly, the high income individuals who are pessimistic early in their life must save most of their funds in the liquid account. They expect their income to immediately decline and remain low throughout the remainder of their working years, and therefore for current savings to be used to smooth consumption during the rest of their working years in addition to retirement. This desire for more liquid savings from high earners is once again not offset by a lower desire from low earners, for similar reasons as before: low earners on average have few liquid assets and are therefore less able to dissave than high earners are to save. Thus, the expectations channel causes liquid savings to be higher in the learning case than the benchmark case; the less pessimistic benchmark individuals can put more of their savings towards retirement, in the illiquid account, because they have to save less for their working years.

As individuals age in both models, they eventually begin to decumulate liquid assets in favor of illiquid, due primarily to the decrease in lifetime uncertainty reducing the precautionary motive. As the necessity of precautionary savings decreases, the need for liquid assets falls as well. Individuals then optimally distribute these funds over their remaining lifetime. Thus, a portion of the funds are put towards the illiquid asset for retirement, some are immediately consumed today, and then the rest are earmarked for future working periods and re-deposited in the liquid account. This decumulation begins slightly earlier in the learning model. This is because much of the lifetime uncertainty is resolved in the first half of the agent’s working life. From that point on, the amount of residual uncertainty in the learning model is not that much higher than in the benchmark model. However, the amount of liquid savings the average individual has accumulated up to that point in the learning model vastly exceeds their benchmark counterpart. Thus the point at which the learning individual no longer needs to expand their liquid holdings occurs a bit sooner than in the benchmark model. Further, because they saved so
Figure 6: Ratio of Illiquid to Liquid Savings

(a) Learning Model

(b) Benchmark Model

much early in their lifetime, learning individuals have a higher stock of savings to reapportion as the precautionary motive falls. They therefore end up with a higher stock of illiquid savings at the end of their working life (as well as higher levels of consumption).

Finally, the expectations channel at the end of the income earning portion of life works in the reverse as compared to at the beginning of life. High income individuals in the learning model are now very optimistic about their future earnings. Thus there is less need for liquid savings, as future working years will be more prosperous than the present and retirement. Thus, these individuals take some of the funds they were saving to bolster consumption during future working periods and divide it between consumption today and retirement. Low income individuals are now very pessimistic, and would like to increase their liquid savings to cover expected future shortfalls in income. However, their current income is low, meaning that they cannot save much in absolute terms. Further, while individuals are able to transfer from the liquid to the illiquid account, they are unable to transfer the opposite way. Therefore, the asymmetry of opportunity causes liquid dissaving and illiquid saving to rise in later periods. Note that this reversal is not present at all in the benchmark model. All individuals still expect mean reversion, and therefore the expectation channel performs the same way throughout the lifecycle: high income individuals have some desire to save in the liquid account and low income individuals would like to dissave.

5.2 Comparison to the Data

Given the very stylized nature of the model, it is precarious to compare too seriously the model to the data. However, it is useful to examine the data to understand the relative successes of each model. I again use data from the 2013 SCF. First, consider average holdings of each asset over the lifecycle. It is clear that, on average, the benchmark model generates too little liquid holdings relative to illiquid
holdings. Consider the peaks of the average holdings for each asset in each model. The benchmark model implies the average individual’s maximum holdings of the illiquid asset are 10.85 times greater than their maximum holdings of the liquid asset. The ratio for the learning model is 2.30, which is much closer to the empirical ratio of 1.82. Further, if one considers the ratio of holdings of illiquid to liquid assets, averaging over the entire lifecycle, it again becomes clear that the benchmark understates holdings of the liquid asset. In order to prevent demographics from biasing the ratio, in the data I first take average holdings of each asset by experience level. I then average all of the experience levels together (in this way each level of experience is weighted the same, regardless of the distribution of the population over experience levels), and then take the ratio. In the data, the average experience level holds 2.63 more illiquid assets than liquid assets. The ratio for the benchmark model is 9.08, while the learning model comes much closer at 1.66.8 Neither model does terribly well matching the shapes of average holdings (though they do capture the single-peakedness of both types of assets).

One object of particular interest is the proportion of assets that individuals choose to hold. Suppose we were calibrating the model to match the total amount of savings. If the composition of those savings depended on the underlying income process, then just by changing the income process we could potentially have different conclusions for the number of liquidity constrained individuals, the relative amount of housing services, or some other metric of interest, even keeping overall savings for each household fixed. I show in Figure 6 that the entire distribution of portfolio composition depends on the income specification. Note that I only plot up until period 40, since the ratio explodes as the amount of liquid assets mechanically approaches zero.

In Figure 6a I plot the 25th percentile, the median, and the 75th percentile of the ratio of illiquid to liquid assets for the learning model. In Figure 6b I plot the same statistics for the benchmark. In both figures this ratio is fairly stable over the lifetime until individuals start decumulating liquid savings as they approach retirement. The main difference is the size of the ratios. For the learning model, the ratio stays at nearly zero until mid-way through the working lifetime. The median individual only holds as much illiquid as liquid assets at 30 years into his working life. This is in contrast to the median benchmark individual, who holds around 3 times as much illiquid as liquid assets throughout most of his lifetime. It is worth noting that these distributions are fairly tight. The 75th percentile of the learning model is below the 25th percentile for the benchmark model throughout the entire lifecycle.

In Figure 7, I compare the median ratio throughout the lifecycle found in the data to its model counterparts. Figure 7a considers both the benchmark and learning model, while 7b ignores the benchmark model to zoom in on the other two series. The benchmark model implies that the median individual holds too many illiquid assets relative to liquid assets throughout the lifecycle. In fact, early in the lifecycle the benchmark model suggests that this ratio is nearly 4, whereas in the data the median individual holds little to no illiquid assets at all. The learning model predicts positive illiquid assets for

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8 Attempting similar comparisons with illiquid holdings and income yield ratios that are way too high in both models. This is partially due to the models having no income in retirement, as well as rates of return that are potentially too high (see Kaplan and Violante (2014)). One result of this is that, to the extent that the learning model gets the ratio of illiquid to liquid savings lower than the benchmark model, the ratio of liquid assets to income over the lifecycle is counterfactually high in the liquid model.
the median individual early in the lifecycle, but this amount is very low relative to liquid assets with a ratio below .5. The data and learning model match well at the end of the lifecycle as well. Where the model does more poorly is mid-age, from around period 10 to 35. Here the curvature is much higher in the learning model: the ratio remains very low until around experience level 20 before slowly increasing in a convex manner. In the data, this ratio increases almost linearly beginning at age 10.

To guard against these results being driven purely by time effects, in the Appendix I plot this series for multiple years. Though this series does have some between year variation, the major conclusions remain the same. Over the entire lifecycle, the benchmark model predicts that the median individual’s illiquid holdings are too high relative to his liquid holdings and the learning model does a good job matching the ratio at the beginning and towards the end of working life. The main differences between years in the data are the point in the lifecycle when the ratio begins to increase and size of the ratio over the middle of the lifecycle. For years other than 2013, this increase begins closer to 5 years of experience than 10. Moreover, the ratio is higher in the middle of the median individual’s life, which makes the difference between the learning model and the data more pronounced and diminishes the difference between the benchmark model and the data. Depending on the year, the benchmark ratio may be closer than the learning ratio for some periods in the middle of the lifecycle. For instance, the benchmark model is only slightly higher than the 2007 ratio for periods surrounding 20 years of experience.

Both models have difficulty matching the entire distribution. As mentioned, both models feature a very tight distribution, which is not the case in the data where the asset choices are more diffuse across percentiles. Thus, we could say that the benchmark model does better “matching” the asset choices of the higher percentiles in the data, while the learning model does better with the lower percentiles, although these matches get worse as the percentiles get further away from the median. In the case

\[9\] It is worth noting that the 75th percentile of the benchmark model does an acceptable job matching the corresponding
of this of the learning model, this deterioration of versimilitude at lower percentiles may be partially mechanical: lower percentiles actually have negative ratios throughout the lifecycle, which is impossible in either model.

5.3 Policy Implications

Clearly, the choice of income process has stark implications for household portfolio decisions over the lifecycle. This could potentially impact the efficacy of fiscal policy. An example of this would be a fiscal stimulus, such as a one-off payment to households. Kaplan and Violante (2014) show that such a one-off payment can generate large consumption changes in a two asset model with liquid and illiquid assets. This is due to the presence of what is termed the “wealthy hand-to-mouth”; that is, individuals who hold positive net worth, yet hold most of it in accounts that are difficult to adjust. These individuals cannot consume out of savings without incurring a fixed cost, and therefore may choose not to dissave even if they would like to increase consumption. These individuals will tend to consume a large portion of a transfer from the government.

This result depends crucially on the number of individuals who are constrained; in other words, the number of individuals who choose to hold low levels of liquid assets. What this paper shows is that switching from the standard approach of modelling income, as was done in Kaplan and Violante (2014), to a plausible alternative can generate higher proportion of liquid savings over the entire lifetime for individuals throughout the distribution. This implies that fewer individuals will be liquidity constrained and will tend to diminish any consumption response to a fiscal transfer. Even if one calibrates parameters to match the total number of hand-to-mouth individuals, this still implies that other parameters of the calibration, such as the coefficient of relative risk aversion, must adjust in response.

Finally, there are potentially other policy implications if one thinks about individual assets, rather than the broad measures used in this paper. For example, suppose one was interested in housing policy, specifically on regulations that tighten or relax down payment constraints. In this environment we can consider the illiquid asset to be housing, which is not a terrible simplification given how much housing dominates household balance sheets, and down payments as a fixed cost, $\gamma_F$, that only applies when $b_{t+1} > b_{\text{min}}$ and $b_t = 0$, for some minimum housing size, $b_{\text{min}}$.

Consider the benchmark model, where the typical individual (both average and median) holds little liquid assets. Introducing a non-zero downpayment will cause many individuals to have to defer illiquid purchases, as they will not have enough saved up to pay for the fixed cost. This will be less of an issue in the learning model, as the typical individual holds more liquid assets and chooses to invest relatively less in the illiquid asset already, due to the mechanisms outlined above. This same reasoning applies to changes in the downpayment. In an enviroment with a downpayment, benchmark agents will increase their liquid savings such that they can afford the downpayment, plus have a little left over. Learning individuals will have a large excess of accessible savings. Thus, for a small increase in $\gamma_F$, many more benchmark agents will be priced out of making an immediate illiquid purchase, likely affecting overall series in the data.
savings decisions more in the benchmark model than in the learning model. Similar policy experiments, such as adjusting government subsidies to retirement savings, may yield different results depending on the assumed income process as well. Future work will consider these situations in detail.

6 Conclusion

When not explicitly examining labor markets, the literature largely assumes that labor income consists solely of a highly persistent component, a transitory component, and occasionally a known idiosyncratic intercept, when incorporating exogenous income into a model. Recent work has shown that income data also supports an alternative specification, where individuals have some idiosyncratic factor that drives income realizations over their lifetime. Using parameters estimated from identical income data, I show that this choice of income process is not trivial when considering household portfolio decisions, where households must decide between investing in a liquid or illiquid asset.

I consider a two asset model where individuals can gradually learn about their future income potential over time. I show that even in a standard, highly stylized model of portfolio choice, these two processes yield drastically different outcomes for household asset decisions. These asset decisions have potential ramifications for various types of policy initiatives. Thus, it is important to understand that an assumption typically unemphasized by the literature, that of the specification of underlying income process, has crucial implications for portfolio choice models and any conclusions drawn from them.

Appendix

Benchmark Model and Computation

Retirement periods are handled identically in the benchmark model as in the model with learning. The differences arise in the working periods, where the model with the benchmark process is more straightforward than the model with learning. Since the individual observes not just his income draw but also the composition of his income, the individual’s beliefs coincide fully with reality, obviating the need for the Kalman Filtering apparatus. However, since each component of income has different implications for the individual, each must be kept track of separately. Thus, there are still five state variables for the following dynamic problem:

\[
V_t(\alpha, z_t, \varepsilon_t, m_t, b_t) = \max_{c_t, b_t, m_t} u(c_t) + \delta E_t[V_{t+1}(\alpha, z_{t+1}, \varepsilon_{t+1}, m_t, b_t)]
\]

In order to simplify the problem computationally, I use a normalization popularized by Carroll (2011) for dynamic programming problems featuring CRRA preferences. When iterating backwards to solve for decision rules, I divide the problem by \(\alpha\). This affects only the levels of assets, consumption, and income (and therefore also the value function), but does not affect the ratios of any of these variables. Thus, this normalization removes a state variable while retaining all important information. When simulating
individual decisions, I draw $\alpha$ for each household separately and then multiply the choice variables (and income) by these draws. This recovers the decisions of households who receive the full income draw, including the individual intercept.

Data

Survey of Consumer Finances (SCF)

The SCF is a triennial survey running from 1983 to 2016. Post-1986 the SCF uses multiple imputations to fill in missing data. Thus, in the raw data there are five records for every household. While averages and frequencies should be unaffected by the implicates, medians may be slightly off. Therefore, for each statistic of interest I follow the advice of the Federal Reserve Board and first calculate that statistic by the implicate, before averaging the five values together. For the 1983 and 1986 surveys only one set of data is provided, and therefore the aforementioned process is unnecessary. Since the SCF oversamples wealthier households, to better capture their more diverse portfolios, I use the provided weights while computing all of the statistics in this paper.

Median Portfolio Comparison Between Years

In Figure 8 I plot the smoothed experience profile of the ratio of illiquid to liquid savings for the median individual for the years 2004, 2007, and 2013. Series from other years are roughly spanned by the three depicted here. Notice that there is a fair bit of variation between the three series, both in size and shape. The 2013 series stays roughly constant at near zero until around ten years experience, where it first becomes concave and then convex. The 2004 and 2007 series begin to increase around an experience level of 5. The 2004 series is convex while the 2007 series begins concave, ends convex, and changes inflection at around 30 years of experience. There are nontrivial differences between the value of the ratio over the lifetime between the three curves, as well. For example, at age 20, the ratio in 2013 is around .5 while the ratio in 2007 is above 2. Further, the median individual has more illiquid net worth
than liquid net worth after 10 years of experience in 2007, compared with nearly 25 years of experience in 2013.

**Derivations**

**Lifetime Income Distributions**

Taking the variance of the general form of the de-meaned income process, we get the following:

\[
\text{Var}(\tilde{y}_{i,e}) = \sigma^2_\alpha + 2\text{cov}_{\alpha\beta}e + \sigma^2_\beta e^2 + \sigma^2_\varepsilon + \sigma^2_\eta \sum_{j=0}^{e-1} \rho^{2j}
\]

Now, of course the sum in the final term depends on the value of \(\rho\), which is typically taken to be between 0 and 1, inclusive. Thus:

\[
\rho = 0 \implies \sigma^2_\alpha + 2\text{cov}_{\alpha\beta}e + \sigma^2_\beta e^2 + \sigma^2_\varepsilon + \sigma^2_\eta
\]

\[
\rho \in (0, 1) \implies = \sigma^2_\alpha + 2\text{cov}_{\alpha\beta}e + \sigma^2_\beta e^2 + \sigma^2_\varepsilon + \sigma^2_\eta [1 - \rho^{2e}]
\]

\[
\rho = 1 \implies = \sigma^2_\alpha + 2\text{cov}_{\alpha\beta}e + \sigma^2_\beta e^2 + \sigma^2_\varepsilon + e\sigma^2_\eta
\]

In the benchmark model, with \(\rho \in (0, 1)\) and \(\beta\) restricted to zero, the variance simplifies:

\[
\sigma^2_\alpha + \sigma^2_\varepsilon + \sigma^2_\eta \frac{1 - \rho^{2e}}{1 - \rho^2}
\]

Notice in Figure 9 that the two profiles have a distinct shape, even with parameters estimated from the same income data. The experience-variance profile is concave in the benchmark parameterization while the learning parameterization yields a profile that is convex after the first few working years. The population-wide variance of income is slightly higher with the benchmark parameters early in the lifecycle, but the variance is much higher later in the lifecycle using the learning estimates.
Kalman Filter

Let \( S_t = \begin{bmatrix} \alpha & \beta & z_t \end{bmatrix}' \). If we let \( F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{bmatrix} \) and \( \nu_t = \begin{bmatrix} 0 & 0 & \eta_t \end{bmatrix}' \), then the law of motion, understood by all agents in the model, is:

\[
S_t = FS_{t-1} + \nu_t
\]

Conditional expectations of the parameters at time \( t \) are denoted by \( \hat{S}_t|t \), while one-period ahead forecasts of the parameters are given by \( \hat{S}_{t+1}|t = F\hat{S}_t|t \). Further, letting \( H = \begin{bmatrix} 1 & t & 1 \end{bmatrix}' \) and \( w_t = \varepsilon_t \), one can write the income process as:

\[
y_t = H'S_t + w_t
\]

Denote \( R = E(\nu_t\nu_t') \) and \( Q = E(w_tw_t') \). It can then be shown that beliefs update according to the following relation:

\[
\hat{S}_t|t = \hat{S}_{t|t-1} + K_t(y_t - H'_tS_{t|t-1})
\]

where \( K_t = P_{t|t-1}H_t(H'_tP_{t|t-1}H_t + R)^{-1} \) is the Kalman Gain.

Finally, the perceived variance-covariance matrix of the parameters of interest evolves according to the following two equations:

\[
P_{t+1|t} = FP_{t|t}F' + Q
\]

\[
P_{t|t} = P_{t|t-1} - P_{t|t-1}H_t(H'_tP_{t|t-1}H_t + R)^{-1}H'_tP_{t|t-1}
\]

Mean Square Error

For the benchmark case, deriving the mean squared error is fairly straightforward:

\[
MSE_{t+s|t} = E_t[(y_{t+s} - y_{t+s|t})^2]
\]

\[
= E_t[(d_{t+s} + x_{t+s} - \varepsilon_{t+s} - d_{t+s} - \rho^sx_t)^2]
\]

\[
= E_t[(\varepsilon_{t+s} + \sum_{j=1}^{s} \rho^{j-1}\eta_{t+s-j})^2]
\]

Since all of these shocks are assumed to be i.i.d., this can be rewritten as:

\[
= \sigma^2 \sum_{j=1}^{s} \rho^{2(j-1)} + \sigma^2_z
\]

\[
= \sigma^2 \frac{1 - (\rho^2)^s}{1 - \rho^2} + \sigma^2_z
\]

32
where the fraction is replaced by 1 if \( \rho = 0 \) an s if \( \rho = 1 \).

The learning case is a little more involved. However, it can be shown that:

\[
MSE_{t+s|t} = H'_{t+s}P_{t+s|t}H_{t+s} + R
\]

where

\[
P_{t+s|t} = F^sP_{t|t}F',s + \sum_{j=0}^{s-1} F^j Q^{F',j}
\]

Note that this error will depend greatly on individual priors, as those will determine \( P_{t|t} \). Of course, the more tight the priors, the less uncertain the individual will be over his entire lifecycle.

References


