On the Failure of the Median Voter Theorem in the Presence of Multiple Contests of Varying Types

Edward Raymond Philpot

Daniel A. Graham, Faculty Advisor

Honors Thesis submitted in partial fulfillment of the requirements for Graduation with Distinction in Economics in Trinity College of Duke University.

Duke University
Durham, North Carolina
April, 2008

Acknowledgments

I would like to thank my advisor, Daniel A. Graham, for all of the helpful suggestions, long discussions, and general guidance that have helped to make this paper possible.

I must also thank my family: my mother, my father, Evan, Diana and Barney for their love, support and encouragement both before and during my time at Duke, it is to them whom I dedicate this work.

Abstract

The median voter theorem, first formalized by Duncan Black in 1948, is the result of a classic model used to describe the positioning of candidates in majority-rule elections, eponymously stating that candidates will converge to the median. The goal of this paper is to describe how the median voter theorem fails to hold in more general cases. Specifically, when multi-contest majority rules elections (such as the United States presidential election) are considered, the median voter theorem fails in the presence of even one winner-take-all constituency; this failure provides opportunities for individual constituencies to skew the equilibrium candidate position toward the position of their median voter.

1 Introduction

The median voter theorem is a powerful result that models behavior in two-candidate elections. This paper explores how, though this result holds in the presence of any combination of contests allocating their votes according to a proportional allocation rule, can fail to hold when even one contest allocating votes according to a winner-take-all rule is included. A failure in median voter theory implies that constituencies holding local elections relevant to larger elections, such as the United States presidential election, may skew results away from the median voter of the total population, toward the median voter of the local election. Generally, constituencies may always exploit this skewing, by allocating votes according to a winner-take-all rule, to make a majority of their constituents at least as well off as they would be under a proportional allocation rule.

2 The Model

Suppose that we have a set of constituencies $i \in \mathcal{I} = \{1, \cdots, I\}$, assigned to which we have a smooth distribution of favorite positions of voters, $d_i(\rho)$, over the unit interval in single-issue space, scaled so that $\int_0^1 d_i(\rho)\ d\rho = c_i$, the proportion of the total population constituency i contains. We add the additional assumption that all

$d_i(\rho)$ are strictly positive on the open support.[1] Suppose further that the combined population distribution of favorite positions is $n(\rho) = \sum_{i \in \mathcal{I}} d_i(\rho)$, with the median denoted M. We index the constituencies so that for $i, j \in \mathcal{I}$, $i < j$ if and only if $m_i \leq m_j$, where m_i denotes the median of the distribution $d_i(\rho)$, and M denotes the median of the combined population $n(\rho)$. Voters vote according to the standard Hotelling model with single-peaked preferences. We suppose that there are two candidates, $j \in \{1, 2\}$ who compete by simultaneously choosing a position, so that $p_j \in [0, 1]$ is candidate j's position in all constituencies $i \in \mathcal{I}$. The candidates do not receive any payoff for their performance in the individual constituencies, though the candidate who gains the largest proportion of the overall population receives a payoff $b > 0$, and ties are settled with both candidates winning with probability $\frac{1}{2}$. We now consider the two different vote allocation rules available to each of the constituencies.

Definition Constituency $i \in \mathcal{I}$, is said to allocate its votes according to a winner-take-all rule if the candidate gaining the majority of constituency i's vote is awarded c_i. We further define the subset $\mathcal{W} \subseteq \mathcal{I}$ to be the subset of all winner-take-all constituencies in $\mathcal{I}$.

Definition Constituency $i \in \mathcal{I}$ is said the allocate its votes according to a proportional assignment if candidates are awarded a proportion of c_i equal to the proportion of the vote received in constituency i. We further define the subset $\mathcal{P} \subseteq \mathcal{I}$ to be the subset of all constituencies utilizing a proportional assignment rule in $\mathcal{I}$.

For reference, an adaptation of the median voter theorem (Black, 1948, and Osbourne, 2004, pp. 261):

Median Voter Theorem. *For a two-candidate contest occurring in single-issue*

[1] While this is not a strict necessity, it allows for the median to be uniquely defined.

space with majority rules and single-peaked preferences, both candidates will assume the position of the median voter in equilibrium.

Example 1 Suppose that we have a population uniformly distributed with $n(\rho) = 1_{\{\rho\in[0,1]\}}$ and a winner-take-all constituency with distribution taken from that of $d_w(\rho) = \frac{\rho}{2} \cdot 1_{\{\rho\in[0,1]\}}$, implying that the remaining, proportional constituency will have distribution $d_p(\rho) = \frac{2-\rho}{2} \cdot 1_{\{\rho\in[0,1]\}}$. We seek the Nash equilibrium position in the two-candidate voting contest for this arrangement.

According to the median voter theorem, the median position in the population, $M = \frac{1}{2}$ would be the Nash equilibrium, however this is not the case. Instead, the pure strategy Nash equilibrium candidate position is $\pi^* = 2 - \sqrt{2} \approx 0.586$. We come upon this solution by considering the effects of capturing the entire proportion of votes assigned to constituency w, in this case $c_w = \frac{1}{4}$. If candidate 1 is located at $p_1 = 2 - \sqrt{2}$, she will capture at least $\int_0^{p_1} d_p(\rho)\, d\rho = \frac{1}{2}$ if candidate 2 is on $(p_1, 1]$. If candidate 2 is on $[0, p_1)$, candidate 1 will capture a proportion of the overall vote of at least $c_w + \int_{\rho_1}^{0} d_p(\rho)\, d\rho = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. As a simple majority of the vote is ensured by assuming the position $p = 2 - \sqrt{2}$, symmetry implies that this will be the unique best reply for both candidates, and therefore the Nash equilibrium in the combined contest.

The above example violates the median voter theorem, and we posit that it is part of a larger class of examples that also lead to Nash equilibria which are skewed from the median.

Proposition 2.1. *If* $\mathcal{I}= \{w, p\}$, $w \in \mathcal{W}$ *and* $p \in \mathcal{P}$ *are winner-take-all and proportional allocation contests, respectively,* $m_w \neq M$, *and* $\int_0^M d_w(\rho)\, d\rho < c_w < \frac{1}{2}$, *then* M *is not a Nash equilibrium position of a winning candidate.*

Proof. See appendix.

We see immediately from proposition 2.1 that the Nash equilibrium position of the candidates as predicted by the median voter theorem, M, does not necessarily hold as a Nash equilibrium when $\#(\mathcal{I}) > 1$. Specifically, if we apply proposition 2.1 to our motivating example, we see that $M = \frac{1}{2}$, $m_w = \frac{1}{\sqrt{2}}$ and that $\int_0^M d_w(\rho)\, d\rho = \frac{1}{16} < \frac{1}{4} = c_w$, so it must be that M is not a Nash equilibrium position for the candidates. This conclusion motivates more general conditions under which the median voter theorem does not apply.

Proposition 2.2. *M, the median of $n(\rho)$, is not a Nash equilibrium if and only if*

$$\sum_{\{i \in \mathcal{W} : m_i < M\}} c_i + \int_0^M \sum_{i \in \mathcal{P}} d_i(\rho)\, d\rho \leq \frac{1}{2}$$

and

$$\sum_{\{i \in \mathcal{W} : m_i > M\}} c_i + \int_M^1 \sum_{i \in \mathcal{P}} d_i(\rho)\, d\rho \leq \frac{1}{2}.$$

Proof. See appendix.

Proposition 2.2 shows that the median voter theorem only holds for a subset of all elections consisting of finite combinations of winner-take-all and proportional allocation constituencies. Specifically, the median voter theorem may not always correctly model elections using anything but a combination of proportional allocation constituencies. An commonly cited example of median voter behavior is the United States presidential election wherein representation in an electoral college determined through winner-take-all elections are used to formally name a president, however proposition 2.1 shows that such a contest may be very sensitive to the presence of the individual winner-take-all votes, and so the population median may, in fact, not be the equilibrium candidate position.

3 Equilibria

Now that we have considered a simple example, and conditions under which the standard median voter result fails to hold, we are interested in establishing a more general Nash equilibrium for combined contests.

Proposition 3.1. *For one winner-take-all allocation constituency with distribution* $d_w(\rho)$*, population proportion* c_w*, and the remainder of the population* $c_p = 1 - c_w$ *allocated according to a proportional allocation implying a distribution of* $d_p(\rho) = n(\rho) - d_w(\rho)$*, the Nash equilibrium position in the two candidate contest is*

$$\pi^* = \begin{cases} \underline{l} & m_w < \underline{l} \\ m_w & \underline{l} \leq m_w \leq \bar{l} \\ \bar{l} & m_w > \bar{l} \end{cases},$$

where

$$\underline{l} = \inf\{p : \int_0^p n(\rho) - d_w(\rho)\, d\rho \geq \frac{1}{2} - c_w\},$$

and

$$\bar{l} = \sup\{p : \int_p^1 n(\rho) - d_w(\rho)\, d\rho \leq \frac{1}{2} - c_w\}.$$

Proof. See appendix.

Proposition 3.1 is our most basic description of the Nash equilibrium arising from the presence of both winner-take-all and proportional allocation contests. We note that it has is a very nice description in terms of the cumulative distributions (denoted by caps, so $\int_0^p f(\rho)\, d\rho = F(p)$, for example). We see

$$\begin{aligned} \underline{l} &= \inf\{p : \int_0^p n(\rho) - d_w(\rho)\, d\rho \geq \frac{1}{2} - c_w\} \\ &= \inf\{p : \int_0^p n(\rho)\, d\rho + c_w \geq \int_0^p d_w(\rho)\, d\rho + \frac{1}{2}\} \\ &= \inf\{p : N(p) + c_w \geq D_w(p) + \frac{1}{2}\}, \end{aligned}$$

and similarly that,

$$\begin{aligned}
\bar{l} &= \sup\{p : \int_p^1 n(\rho) - d_w(\rho)\, d\rho \geq \frac{1}{2} - c_w\} \\
&= \sup\{p : 1 - c_w - \int_0^p n(\rho) - d_w(\rho) d\rho \geq \frac{1}{2} - c_w\} \\
&= \sup\{p : \frac{1}{2} + \int_0^p d_w(\rho)\, d\rho \geq \int_0^p n(\rho) d\rho\}. \\
&= \sup\{p : \frac{1}{2} + D_w(p) \geq N(p)\},
\end{aligned}$$

implying that $\underline{l}$ is the position furthest to the left a candidate who captures the winner-take-all vote allocation, c_w, may assume and still have captured the votes of one-half of the total population, and that $\bar{l}$ is the rightmost position a candidate who captures the winner-take-all vote allocation, c_w, may take and still have captured the votes of one-half of the total population.

Just for reference, recall that

$$\begin{aligned}
m_w &= \{p : \int_0^p d_w(\rho)\, d\rho = \frac{c_w}{2}\} \\
&= \{p : D_w(p) = \frac{c_w}{2}\},
\end{aligned}$$

so if we consider our motivating example, we see that $N(\rho) = \rho \cdot 1_{\{\rho \in [0,1]\}}$ as $n(\rho) = 1_{\{\rho \in [0,1]\}}$, and $D_w(\rho) = \frac{\rho^2}{4} \cdot 1_{\{\rho \in [0,1]\}}$ as $d_w = \frac{\rho}{2} \cdot 1_{\{\rho \in [0,1]\}}$, which, in turn, implies that

$$\begin{aligned}
\underline{l} &= \inf\{p : N(p) + c_w \geq D_w(p) + \frac{1}{2}\} \\
&= \{p \in [0,1] : \rho + \frac{1}{4} = \frac{\rho^2}{4} + \frac{1}{2}\},
\end{aligned}$$

$$\begin{aligned}
\bar{l} &= \sup\{p : D_w(p) + \frac{1}{2} \leq N(p)\} \\
&= \{p \in [0,1] : \frac{\rho^2}{4} + \frac{1}{2} = \rho - \frac{1}{4}\}
\end{aligned}$$

and $m_w = \{p \in [0,1] : \frac{\rho^2}{4} = \frac{1}{8}\}$, all displayed in Figure 1 (see appendix B). $\underline{l} = 2 - \sqrt{3} \approx 0.268$, $m_w = \frac{1}{\sqrt{2}} \approx 0.707$ and $\bar{l} = 2 - \sqrt{2} \approx 0.586$, so $\underline{l} < \bar{l} < m_w$, so the Nash equilibrium of the two-candidate game will be $\bar{l}$ according to proposition 3.1.

Proposition 3.1 is very limited in scope as it only tells us what the equilibrium position of two candidates is in the event that there is only one winner-take-all constituency with the remainder of the population's vote being assigned to candidates according to a proportional allocation rule. We now consider a more general description of the equilibrium candidate position when our total population is allocated to candidates according to any finite combination of winner-take-all and proportional allocation contests.

Proposition 3.2. *For any finite combination of contests utilizing winner-take-all and proportional allocation rules, and two additional, vacant winner-take-all constituencies* $\{0, I+1\} \subset \mathcal{W}$ *so that* $m_0 = 0$, $m_{I+1} = 1$, *and* $c_0, c_{I+1} = 0$,

$$\begin{aligned}
\pi^* &= \min_{i \in \mathcal{W}} \{\inf\{p \geq m_i : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\, d\rho + \sum_{\{j \in \mathcal{W} : j \leq i\}} c_j \geq \frac{1}{2}\}\} \\
&= \max_{i \in \mathcal{W}} \{\sup\{p \leq m_i : \int_p^1 n(\rho) - \sum_{k \in \mathcal{W}} d_k(\rho)\, d\rho + \sum_{\{j \in \mathcal{W} : j \geq i\}} c_j \geq \frac{1}{2}\}\}
\end{aligned}$$

is the pure strategy Nash equilibrium position for two-candidates.

Proof. See appendix.

Proposition 3.2 provides a very general description of the Nash equilibrium candidate position, though there is simple intuition behind it. We see that

$$\begin{aligned}
\pi^* &= \min_{i\in\mathcal{W}}\{\inf\{p \geq m_i : \int_0^p n(\rho) - \sum_{i\in\mathcal{W}} d_i(\rho)\; d\rho + \sum_{\{j\in\mathcal{W}:j\leq i\}} c_j \geq \frac{1}{2}\}\} \\
&= \min_{i\in\mathcal{W}}\{\inf\{p \geq m_i : N(p) - \sum_{i\in\mathcal{W}} D_i(p) + \sum_{\{j\in\mathcal{W}:j\leq i\}} c_j \geq \frac{1}{2}\}\} \\
&= \min_{i\in\mathcal{W}}\{\max\{m_i, \inf\{p : N(p) - \sum_{i\in\mathcal{W}} D_i(p) + \sum_{\{j\in\mathcal{W}:j\leq i\}} c_j \geq \frac{1}{2}\}\}\},
\end{aligned}$$

and that the terms in the set

$$K := \{\inf\{p : N(p) - \sum_{i\in\mathcal{W}} D_i(p) + \sum_{\{j\in\mathcal{W}:j\leq i\}} c_j \geq \frac{1}{2}\}\}_{i\in\mathcal{W}} \tag{1}$$

are very similar to the term $\underline{l} = \inf\{p : N(p) + c_w \geq D_w(p) + \frac{1}{2}\}$ from proposition 3.1. Recall that $\underline{l}$ the leftmost position that a candidate may assume given that he has captured the winner-take-all vote allocation, while still being allocated at least one-half of the overall vote. In a similar manner, we see that $k_i \in K$ is the leftmost position that a candidate who has been been given the vote allocations of the winner-take-all constituencies j with medians $m_j \leq m_i$ by the indexing of $\mathcal{I}$. Therefore, when we call for $\max\{m_i, k_i\}$, we are insisting that the candidate who relies on only the votes to the left of his position be to the right of any winner-take-all medians whose votes he claims. Given this, we see that $\min_{i\in\mathcal{W}} k_i$ is merely the leftmost of the positions k_i.[2]

Now that we have an idea of the intuition behind this, we see the equilibrium description from proposition 3.2 reduces to the description provided in proposition 3.1in the single winner-take-all constituency case. We have that $\mathcal{W} = \{0, 1, 2\}$, where

[2] This makes intuitive sense because any competing candidate may get arbitrarily close to this candidate from the right.

constituencies $0, 2 \in \mathcal{W}$ are vacant and constituency $1 \in \mathcal{W}$ is the single constituency with $c_1 \geq 0$, so

$$\begin{aligned}
\pi^* &= \min_{i \in \mathcal{W}} \{\inf\{p \geq m_i : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\ d\rho + \sum_{\{j \in \mathcal{W}: j \leq i\}} c_j \geq \frac{1}{2}\}\} \\
&= \min\{\inf\{p : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\ d\rho \geq \frac{1}{2}\}, \\
&\qquad \inf\{p \geq m_1 : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\ d\rho + c_1 \geq \frac{1}{2}\}\} \\
&= \begin{cases} \underline{l} & m_1 \leq \underline{l} \\ m_1 & \underline{l} < m_1 < \bar{l} \\ \bar{l} & \bar{l} < m_1 \end{cases}
\end{aligned}$$

as $\inf\{p \geq m_1 : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\ d\rho + c_1 \geq \frac{1}{2}\} = \min\{m_1, \underline{l}\}$, and $\inf\{p : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\ d\rho \geq \frac{1}{2}\} = \bar{l}$ for $\underline{l}$ and $\bar{l}$ defined in proposition 3.1, which further subsumes the median voter theorem result when the conditions of proposition 2.2 hold.

Example 2 This example uses the same general set-up as example 1 with the exception that the distribution of voters represented in a winner-take-all contest $d_w(\rho) = \frac{\rho}{2} \cdot 1_{\rho \in [0,1]}$ is now subdivided into three winner-take-all constituencies with the equal population proportions $c_1, c_2, c_3 = \frac{1}{12}$ $d_1(\rho) = \frac{\rho}{2} \cdot 1_{\{\rho \in [0, \frac{1}{\sqrt{3}})\}}$, $d_2(\rho) = \frac{\rho}{2} \cdot 1_{\{\rho \in [\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}})\}}$, and $d_3(\rho) = \frac{\rho}{2} \cdot 1_{\{\rho \in [\sqrt{\frac{2}{3}}, 1]\}}$. Therefore, by proposition 3.2, we see that $\pi^* = \min_{i \in \mathcal{W}} \{\inf\{p \geq m_i : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\ d\rho + \sum_{\{j \in \mathcal{W}: j \leq i\}} c_j \geq \frac{1}{2}\}\}$, and so we see that for the set K as defined in equation 1, we have

$$\begin{aligned}
K &= \{\inf\{p : N(p) - \sum_{i \in \mathcal{W}} D_i(p) + \sum_{\{j \in \mathcal{W}: j \leq i\}} c_j \geq \frac{1}{2}\}\}_{i \in \mathcal{W}} \\
&= \{\inf\{p : \rho - \frac{\rho^2}{4} + \sum_{\{j \in \mathcal{W}: j \leq i\}} c_j \geq \frac{1}{2}\}\}_{i \in \mathcal{W}},
\end{aligned}$$

Table 1: Computed Values for Example 2

$i \in \mathcal{W}$	m_i	k_i	$\max\{m_i, k_i\}$
0	0	$2-\sqrt{\frac{6}{3}}$	$2-\sqrt{\frac{6}{3}}$
1	$\sqrt{\frac{1}{6}}$	$2-\sqrt{\frac{7}{3}}$	$2-\sqrt{\frac{7}{3}}$
2	$\sqrt{\frac{3}{6}}$	$2-\sqrt{\frac{8}{3}}$	$\sqrt{\frac{3}{6}}$
3	$\sqrt{\frac{5}{6}}$	$2-\sqrt{\frac{9}{3}}$	$\sqrt{\frac{5}{6}}$
4	1	0	1

whose elements are displayed in table 1, and pictorial description of which we have in figures 2 and 3 (see appendix B). We see from table 1 that $\inf\{\max\{m_i, k_i\}_{i\in\mathcal{W}}\} = 2 - \sqrt{\frac{7}{3}} \approx 0.472$, and is our Nash equilibrium by proposition 3.2. Just as a check, we see that this is, in fact, not the same as the Nash equilibrium candidate position that we found in example 1, $2-\sqrt{2} \approx 0.586$ so we see that equilibria are distinct from each other, and that dividing the winner-take-all population has a significant effect on the equilibrium candidate position.

We have seen from example 2 how the Nash equilibrium candidate position may be found in practice. We now are interested in the properties of the equilibrium point, namely its position in certain cases.

Corollary 3.3. *For a median of the combined distribution, M, if $\sum_{\{j\in\mathcal{W}:m_j\geq M\}} c_j \geq \frac{1}{2}$, or $\sum_{\{j\in\mathcal{W}:m_j\leq M\}} c_j \geq \frac{1}{2}$, then $\pi^* \in \{m_i\}_{i\in\mathcal{W}}$.*

Proof. See appendix.

Corollary 3.3 states that if more than one-half of the total population is represented in winner-take-all constituencies whose medians fall on one side of the median of the combined distribution, M, then the Nash equilibrium position in the two-candidate game will be the median of one of the winner-take-all contests. We note that the above result is a multiple winner-take-all contest analog to the case in proposition 3.1 where $\underline{l} < m_w < \bar{l}$, giving a Nash equilibrium of m_w. Specifically, if we consider a contest consisting mainly of winner-take-all contests, such as the election of members of the electoral college for the United States presidential election, corollary 3.3 in conjunction with the median voter theorem states that the Nash equilibrium position of both candidates (and so the winner) should have assumed the position of the median voter of one of the states.

4 Skewing

Corollary 3.3 raises an interesting question about the role of different combinations of winner-take-all and proportional allocation contests, specifically, we would like to determine how far it is possible for the Nash equilibrium candidate to be skewed from the median of the combined populations, M. We are capable of finding this result exactly (it is given in proposition 3.2) given perfect information including the exact combination of winner-take-all and proportional allocation constituencies and their distributions, though in the absence of this kind of knowledge we would like to determine boundaries on equilibrium behavior given a proportion of the population represented in contests of both types.

Lemma 4.1. *If we have a distribution $d_0(\rho)$ with median M, and set of distributions $\{d_i(\rho)\}_{i\in\mathcal{K}}$ so that $\sum_{i\in\mathcal{K}} d_i(\rho) = d_0(\rho)$, then if the median of a distribution $d_i(\rho)$ for $i \in \mathcal{K}$ is denoted m_i, there exists at least one $j \in \mathcal{K}$ so that $m_j \geq M$.*

Proof. See appendix.

Proposition 4.2. *If $|\pi^* - M|$ is at a maximum given $c_w < \frac{1}{2}$, then either $m_i = m_j$ for all $i, j \in \mathcal{W}$ or $\sum_{i\in\mathcal{W}} d_i(\rho)$ is taken to be a single winner-take-all constituency.*

Proof. See appendix.

The purpose of proposition 4.2 is to allow us to simplify our analysis of the boundaries of the equilibrium behavior by stating that, given a proportion of the population represented by winner-take-all contests, for the boundaries on equilibrium behavior to be the largest, we must consider the winner-take-all constituencies as one. At first glance, it may appear that proposition 4.2 is weakened by the restriction on c_w. However, we note that this is necessary for if $c_w > \frac{1}{2}$ then $\pi^* = m_w$, by corollary 3.3 and the median voter theorem.

Proposition 4.3. *Given a distribution of the total population $n(\rho)$ with cumulative distribution function $N(\rho) = \int_0^\rho n(\rho)\ d\rho$ and a proportion of the total population $c_w < \frac{1}{2}$ allocated according to a winner-take-all rule, the lower bound Nash equilibrium position is $\underline{l}^*(c_w) = N^{-1}\left(\frac{1-c_w}{2}\right)$, and the upper bound Nash equilibrium position is $\overline{l^*}(c_w) = N^{-1}\left(\frac{1+c_w}{2}\right)$.*

Proof. See appendix.

Proposition 4.3 states that the furthest that the equilibrium candidate position may be skewed from the median in any case where $c_w < 1/2$ is, according to proposition 3.1 $\max\{M - \underline{l}, \bar{l} - M\}$, and this only occurs when $\pi^* = \underline{l}$ for $\underline{l}$ at a minimum, or $\pi^* = \bar{l}$ for $\bar{l}$ at a maximum. $\pi^* = \underline{l}, \bar{l}$ only when $m_w \in [0, \underline{l}] \cup [\bar{l}, 1]$, though we note that if $m_w < \underline{l}$, then it must be the case that $\underline{l}$ is not at a minimum as $d_w(\rho) > 0$ and $n(\rho) > 0$ on the open support, and intuitively $\underline{l}$ occurs at the intersection of $D_w(\rho) + \frac{1}{2}$ and $N(\rho) + c_w$. Therefore, the least $\underline{l}$ can be when $\underline{l} = m_w$, and similarly, the greatest that $\bar{l}$ can be is when $\bar{l} = m_w$, which occur at $N^{-1}\left(\frac{1-c_w}{2}\right)$ and $N^{-1}\left(\frac{1+c_w}{2}\right)$, respectively.

We have avoided treating the $c_w > \frac{1}{2}$ case until this point because it introduces another issue, specifically, the question of what constitutes a constructible median.

We note that mass in the tails of our combined distributions is bounded, so if we attempt to cut a winner-take-all distribution of a given size, c_w, from it, we see that the median must be within the interval $[\underline{m}(c_w), \overline{m}(c_w)] = [N^{-1}\left(\frac{c_w}{2}\right), N^{-1}\left(\frac{2-c_w}{2}\right)]$, for if the median of a winner-take-all distribution with a proportion of the vote c_w is at a point m, it must be that $N(m) \geq \frac{c_w}{2}$ and $1 - N(m) \geq \frac{c_w}{2}$. This implies that in order for the Nash equilibrium to be skewed as as far from the median of the population as possible, it must be that m_w be both constructible, and equal to either $\underline{l}$ or $\bar{l}$. This implies that π^* is on the boundary of $[\underline{l^*}(c_w), \overline{l^*}(c_w)] \cap [\underline{m}(c_w), \overline{m}(c_w)]$ when it maximally skewed. By construction, we see that $[\underline{l^*}(c_w), \overline{l^*}(c_w)] \subseteq [\underline{m}(c_w), \overline{m}(c_w)]$ for $c_w < \frac{1}{2}$, and similarly that $[\underline{l^*}(c_w), \overline{l^*}(c_w)] \supseteq [\underline{m}(c_w), \overline{m}(c_w)]$ for $c_w \geq \frac{1}{2}$.[3] Further $\overline{m}(c_w) - \underline{m}(c_w)$ is decreasing in c_w, and $\overline{l^*}(c_w) - \underline{l^*}(c_w)$ is increasing in c_w.

When we apply proposition 4.3 and the constructible median concept to the general set-up from example 1, that is $N(\rho) = \rho \cdot 1_{\{\rho \in [0,1]\}}$, we see that we have

$$\begin{aligned}
[\underline{l^*}(c_w), \overline{l^*}(c_w)] &= [N^{-1}\left(\frac{1-c_w}{2}\right), N^{-1}\left(\frac{1+c_w}{2}\right)] \\
&= [\frac{1-c_w}{2}, \frac{1+c_w}{2}] \\
&= [1-2\rho, 2\rho-1],
\end{aligned}$$

and

$$\begin{aligned}
[\underline{m}(c_w), \overline{m}(c_w)] &= [N^{-1}\left(\frac{c_w}{2}\right), N^{-1}\left(\frac{2-c_w}{2}\right)] \\
&= [\frac{c_w}{2}, \frac{2-c_w}{2}] \\
&= [2\rho, 2-2\rho],
\end{aligned}$$

[3] As $\underline{m} = N^{-1}\left(\frac{c_w}{2}\right)$ and $\underline{l} = N^{-1}\left(\frac{1-c_w}{2}\right)$, we see that $\underline{l} \leq \underline{m}$ if and only if $N^{-1}\left(\frac{1-c_w}{2}\right) \leq N^{-1}\left(\frac{c_w}{2}\right)$ if and only if $\frac{1-c_w}{2} \leq \frac{c_w}{2}$ if and only if $\frac{1}{2} \leq c_w$, and similarly $\overline{l^*}(c_w) \geq \overline{m}(c_w)$ if and only if $c_w \geq \frac{1}{2}$.

plotted in figure 4 (see appendix B). What these boundaries tell us is the greatest extent to which the Nash equilibrium candidate position may be skewed from the median of the combined population M given nothing but the size of the proportion of the population represented by winner-take-all contests. We take caution to note, however, that the constructible median is only guaranteed to hold for a single winner-take-all constituency, though it is a consequence of these boundaries that the furthest skewed positions, under any circumstance, that Nash equilibrium candidate position may occupy are $\{\underline{m}(\frac{1}{2}), \overline{m}(\frac{1}{2})\}$ (see figure 4 in appendix B).

5 Implications for Constituencies

From the above, we see that constituencies have a certain degree of power over the outcome of an election depending on whether or not they employ a winner-take-all allocation scheme. We are led to ask under what circumstances would a constituency decide to employ such a rule.

Proposition 5.1. *Given a constituency $g \in \mathcal{I}$ with distribution $d_g(\rho) \leq n(\rho)$, if we denote the Nash equilibrium position in the two-candidate game when $g \in \mathcal{W}$ as π^*_{WTA}, and π^*_P when $g \in \mathcal{P}$, all else equal, then*

$$|\pi^*_{WTA} - m_g| \leq |\pi^*_P - m_g|.$$

Proof. See appendix.

Corollary 5.2. *If a simple majority referendum is the mechanism used to determine whether constituency $g \in \mathcal{I}$ allocates its votes according to a winner-take-all or a proportional allocation scheme, then constituency g will always allocate its votes according to a winner-take-all rule.*

Proof. As we have assumed that the mechanism is a majority rules referendum, the median voter theorem applies, implying the result by proposition 5.1.

Proposition 5.1 and corollary 5.2 imply that $\mathcal{I} = \mathcal{W}$ is a pure strategy Nash equilibrium with constituencies acting as agents because at least majority of the constituents in a constituency deciding between holding a winner-take-all and proportional allocation contest would be better off by choosing winner-take-all. This is implied by the skewing results that have been developed so far. We note that this equilibrium is not necessarily unique, however, for there may exist at least one constituency $j \in \mathcal{I}$ for which $|\pi^*_{WTA} - m_j| = |\pi^*_P - m_j|$ as defined in proposition 5.1 implying that the voters in j are (in an aggregate sense) indifferent between winner-take-all and proportional allocations.

6 Conclusion and Possible Extensions

We have seen that, while the median voter theorem gives a general feel for the manner in which candidates compete to gain a simple majority of the population, it also omits details that allow for a small portion of the population to have influence over the final results of combined contests. Additionally, we see that the median voter result by itself does not show why constituencies assigning delegates in a United States presidential primary would be more often assigning delegates according to winner-take-all allocation (in the absence of regulations forbidding it).

There are a number of possible directions for future research in this area, a particularly ambitious extension is to consider cases including more than two candidates. There are also possible refinements to the boundaries found in the section 4 when we include information about the number of winner-take-all constituencies, their individual sizes, and restrictions on how the distributions of these individual constituencies over single-issue space differ from the overall population.

Appendix

A Proposition Proofs

Proof of Proposition 2.1. We proceed with a proof by contradiction, so suppose that M is a Nash equilibrium position for the two candidates and that, without loss of generality, $m_w < M$. As M is a Nash equilibrium by supposition, $\int_0^M d_p(\rho)\ d\rho + d_w = 1/2$, but

$$\begin{aligned}
\int_0^M d_p(\rho)\ d\rho + d_w &= \int_0^M n(\rho)\ d\rho - \int_0^M d_w(\rho)\ d\rho + d_w \\
&= \int_0^M n(\rho)\ d\rho - \int_0^M d_w(\rho)\ d\rho + \int_0^M d_w(\rho)\ d\rho + \int_M^1 d_w(\rho)\ d\rho \\
&= \int_0^M n(\rho)\ d\rho + \int_M^1 d_w(\rho)\ d\rho \\
&> 1/2
\end{aligned}$$

as $\int_M^1 d_w(\rho)\ d\rho > 0$ by hypothesis, a contradiction. This implies that one of the candidates may deviate to some $p' < M$ so that $|p' - m_w| < |M - m_w|$ and $\int_0^{p'} d_p(\rho)\ d\rho + d_w \geq \frac{1}{2}$, and thus that the candidate at position p' will win. □

Proof of Proposition 2.2. M, the median of $n(\rho)$ is a Nash equilibrium if and only if a candidate may not advantageously deviate from M, which occurs if and only if all other positions, given a candidate at M receive a proportion of the vote less than one-half. We exhaust the possibilities.

Suppose a candidate deviates to $p < M$. We therefore see that the deviating candidate shall receive $\int_0^{\frac{p+M}{2}} \sum_{i \in \mathcal{P}} d_i(\rho)\ d\rho + \sum_{\{i \in \mathcal{W}: m_i \leq \frac{p+m}{2}\}} c_i \leq \int_0^M \sum_{i \in \mathcal{P}} d_i(\rho)\ d\rho + \sum_{\{i \in \mathcal{W}: m_i < M\}} c_i \leq \frac{1}{2}$. This implies that if $\int_0^M \sum_{i \in \mathcal{P}} d_i(\rho)\ d\rho + \sum_{\{i \in \mathcal{W}: m_i < M\}} c_i < \frac{1}{2}$, then the candidate at M now has a proportion of the vote $1 - \int_0^{\frac{p+M}{2}} \sum_{i \in \mathcal{P}} d_i(\rho)\ d\rho +$

$\sum_{\{i\in\mathcal{W}:m_i\leq\frac{p+m}{2}\}} c_i > \frac{1}{2}$. However, if $\int_0^M \sum_{i\in\mathcal{P}} d_i(\rho)\ d\rho + \sum_{\{i\in\mathcal{W}:m_i<M\}} c_i = \frac{1}{2}$, then we see that $\int_M^1 \sum_{i\in\mathcal{P}} d_i(\rho)\ d\rho + \sum_{\{i\in\mathcal{W}:m_i>M\}} c_i = \frac{1}{2}$, and either there is a tie, or the candidate at M has won. Therefore, there was no advantageous deviation, implying that M is a Nash equilibrium candidate position. $\square$

Proof of Proposition 3.1. We proceed case wise. Suppose that $m_w < \underline{l}$ as defined above. We see that a candidate positioned at $\underline{l}$ receives $\frac{1}{2} - c_w$ of the vote from the portion of the population whose vote is allocated proportionally, and c_w from the winner-take-all population with median $m_w < \underline{l}$ if the competing candidate enters at some $v > \underline{l}$, so the candidate at $\underline{l}$ receives at least $\frac{1}{2} = \frac{1}{2} - c_w + c_w$ of the vote. Suppose that the competitor enters at some $p < \underline{l}$, then the candidate at $\underline{l}$ gains some allocation $a < \frac{1}{2} - c_w + c_w = \frac{1}{2}$ implying that the candidate at $\underline{l}$ will receive a portion of the vote $1 - a > \frac{1}{2}$, therefore it is both candidates' best reply to set their position as $\underline{l}$ when $m_w < \underline{l}$.

Suppose that $m_w > \bar{l}$, then $\bar{l}$ is the Nash equilibrium by the same logic as the $m_w < \underline{l}$ case.

Suppose that $\underline{l} < m_w < \bar{l}$, then the Nash equilibrium position of both candidates is m_w, for if one candidate is locates herself at m_w, and another locates herself at a position $p < m_w$, then we see that the candidate at m_w gains a portion of the vote c_w from being closest to m_w, and as $p < m_w < \bar{l}$, we see that the candidate at m_w gains at least $\frac{1}{2} - c_w$ from the proportionally allocated population to the right of $\bar{l}$, so the candidate at m_w gains a proportion of the population greater than $\frac{1}{2}$ and therefore wins. The logic is the same if the deviating candidate chooses some position $p > m_w$. $\square$

Proof of Proposition 3.2. We proceed with a proof by contradiction. Suppose that some $p > \pi^*$ is winning position. This implies that for

$$\sum_{\{j \in \mathcal{W}: m_j \geq p\}} c_j + \int_p^1 n(\rho) - \sum_{\{i \in \mathcal{W}\}} d_i(\rho)\, d\rho \geq \frac{1}{2},$$

a contradiction by the definition of π^*.

Suppose that some $p < \pi^*$ is a winning position. This implies that

$$\sum_{\{j \in \mathcal{W}: m_j \leq p\}} c_j + \int_0^p n(\rho) - \sum_{\{i \in \mathcal{W}\}} d_i(\rho)\, d\rho \geq \frac{1}{2},$$

a contradiction by the minimality of π^*. We therefore see that there is no advantageous deviation from π^* implying that it is a Nash equilibrium. □

Proof of Corollary 3.3. Without loss of generality, let $\underline{k} = \arg\min\{i \in \mathcal{W} : \sum_{j \leq i} c_j \geq \frac{1}{2}\}$ and $m_{\underline{k}} \leq M$. By the definition of $m_{\underline{k}}$, we see that to the left of M we have a proportion of at least $\frac{1}{4}$ of the total population represented by winner-take-all constituencies, leaving no more than $\frac{1}{4}$ to be represented by proportional allocation constituencies to the left of M. This implies that, as $m_{\underline{k}} \leq M$, $\int_0^{m_{\underline{k}}} n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\, d\rho \leq \frac{1}{4}$. By proposition 3.2, we see that the Nash equilibrium will be $m_{\underline{k} \in \{m_i\}_{i \in \mathcal{W}}}$. □

Proof of Lemma 4.1. We proceed with a proof by contradiction. Suppose that for all $j \in \mathcal{K}$, $m_j < M$. Then we may find $\overline{m} = \max_{i \in \mathcal{K}}\{m_i\}$ so that $\overline{m} < M$. This implies that

$$\int_0^{\overline{m}} \sum_{i \in \mathcal{K}} d_i(\rho)\, d\rho = \int_0^{\overline{m}} d_0(\rho)\, d\rho \geq \frac{1}{2},$$

a contradiction by the definition of M and the fact that it is uniquely by supposition.

□

Proof of Proposition 4.2. We proceed with a proof by contradiction. Without loss of generality, let $m_w < M$. Suppose that $\max |\pi^* - M|$ does not occur when we consider all of the winner-take-all contests as a single contest with a combined distribution. Proposition 3.1 states that the Nash equilibrium for this case, $\pi_c^* \geq m_w$ in all cases. Similarly, proposition 3.2 states that the Nash equilibrium for the case in which the winner-take-all mass is subdivided is

$$\pi_d^* = \min_{i \in \mathcal{W}} \{\inf\{p \geq m_i : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\, d\rho + \sum_{\{j \in \mathcal{W}: j \leq i\}} c_j \geq \frac{1}{2}\}\}.$$

We have, therefore, assumed that $\pi_d^* < \pi_c^*$, so it must be that

$$\int_0^{\pi_d^*} n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\, d\rho + \sum_{\{j \in \mathcal{W}: m_j < \pi_d^*\}} c_j \geq \frac{1}{2},$$

a contradiction. Lemma 4.1 says that for some $j \in \mathcal{W}$, $m_j \geq M$, so $\sum_{\{i \in \mathcal{W}: m_i \leq \pi_d^*\}} c_i \leq \sum_{j \in \mathcal{W}} c_j$ This implies that if $\sum_{\{j \in \mathcal{W}: m_j \leq \pi_d^*\}} c_j = \sum_{i \in \mathcal{W}} c_i$, then we see that $M < \pi_d^* < \pi_c^*$ and $\int_0^{\pi_d^*} n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\, d\rho + \sum_{j \in \mathcal{W}} c_j \geq \frac{1}{2}$, a contradiction by the minimality of π_c^*. If $\sum_{\{j \in \mathcal{W}: m_j \leq \pi_d^*\}} c_j < \sum_{i \in \mathcal{W}} c_i$, then we see that

$$\int_0^{\pi_d^*} n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\, d\rho + \sum_{\{j \in \mathcal{W}: m_j \leq \pi_d^*\}} c_j \leq \int_0^{\pi_c^*} n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\, d\rho + \sum_{\{j \in \mathcal{W}: m_j \leq \pi_d^*\}} c_j < \frac{1}{2}$$

by the definition of π_c^*. We therefore see that it must be that $\pi_d^* \geq \pi_c^*$. □

Proof of Proposition 4.3. Proposition 3.1 gives us the Nash equilibrium position of in the two-candidate game when we have only one winner-take-all constituency, so we use this result here. We note that the Nash equilibrium has lower bound

$$\underline{l} = \inf\{p : \int_0^p n(\rho) - d_w(\rho)\, d\rho \geq \frac{1}{2} - c_w\},$$

and upper bound

$$\bar{l} = \sup\{p : \int_p^1 n(\rho) - d_w(\rho)\, d\rho \leq \frac{1}{2} - c_w\},$$

though we note that these only become active restrictions when $m_w \leq \underline{l}$ and $m_w \geq \bar{l}$, and as $m_w = \{p : N(p) = \frac{c_w}{2}\}$, it must be that this happens when $\underline{l^*} = N^{-1}\left(\frac{1-c_w}{2}\right)$. We see by a similar argument that $\overline{l^*} = N^{-1}\left(\frac{1+c_w}{2}\right)$. Just as an example, we are able to construct distributions $\underline{d}_w(\rho), \overline{d}_w(\rho) \leq n(\rho)$ so that $m_w = \underline{l}$ or $m_w = \bar{l}$, respectively. These are

$$\underline{d}_w = \begin{cases} \frac{c_w n(\rho)}{1-c_w} & 0 \leq \rho \leq N^{-1}\left(\frac{1-c_w}{2}\right) \\ \frac{c_w n(\rho)}{2(1+c_w)} & N^{-1}\left(\frac{1-c_w}{2}\right) < \rho \leq 1 \end{cases},$$

and

$$\overline{d}_w = \begin{cases} \frac{c_w n(\rho)}{1+c_w} & 0 \leq \rho \leq N^{-1}\left(\frac{1+c_w}{2}\right) \\ \frac{c_w n(\rho)}{2(1-c_w)} & N^{-1}\left(\frac{1+c_w}{2}\right) < \rho \leq 1 \end{cases},$$

which work for most non-pathological distributions $n(\rho)$. □

Proof of Proposition 5.1. Suppose without loss of generality that $m_g \leq M$, and we proceed case wise. By proposition 3.2 we have that the Nash equilibrium position of the candidates is

$$\min_{i\in\mathcal{W}}\{\inf\{p \geq m_i : \int_0^p n(\rho) - \sum_{i\in\mathcal{W}} d_i(\rho)\, d\rho + \sum_{\{j\in\mathcal{W}: j\leq i\}} c_j \geq \frac{1}{2}\}\}.$$

Suppose that $\pi^*_{WTA} < m_g$, denote the index set of winner-take-all constituencies as $\mathcal{W}_1$ after g changes to proportional allocation and $\mathcal{W}_0$ before. We therefore see that

$$\int_0^{\pi^*_{WTA}} n(\rho) - \sum_{i\in\mathcal{W}_1} d_i(\rho)\, d\rho \geq \int_0^{\pi^*_{WTA}} n(\rho) - \sum_{i\in\mathcal{W}_0} d_i(\rho)\, d\rho,$$

and as $\pi^*_{WTA} < m_g$, we have that $\sum_{\{j\in\mathcal{W}_0: j\leq\pi^*_{WTA}\}} c_j = \sum_{\{j\in\mathcal{W}_1: j\leq\pi^*_{WTA}\}} c_j$, so $\pi^*_P \leq \pi^*_{WTA}$.

Suppose that $\pi^*_{WTA} > m_g$. We note that proposition 3.2 allows for the alternative specification of the equilibrium as

$$\max_{i\in\mathcal{W}}\{\sup\{p \leq m_i : \int_p^1 n(\rho) - \sum_{k\in\mathcal{W}} d_k(\rho)\, d\rho + \sum_{\{j\in\mathcal{W}:j\geq i\}} c_j \geq \frac{1}{2}\}\}.$$

By logic similar to the $\pi^*_{WTA} < m_g$ case, this result holds. $\square$

B Figures

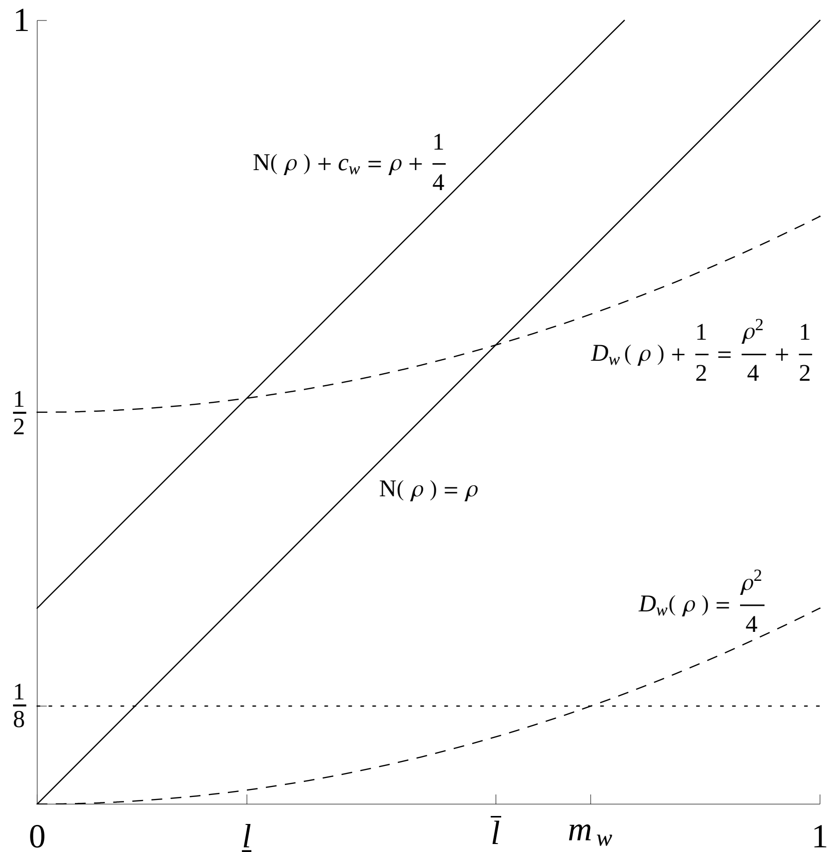

Figure 1: Above we see how the values $\underline{l}$, $\bar{l}$, and m_w may be derived from the cumulative distribution functions of $n(\rho)$ and $d_w(\rho)$ (taken from our motivating example). We see that $\underline{l} < \bar{l} < m_w$ implying that $\bar{l}$ is the Nash equilibrium position by proposition 3.1.

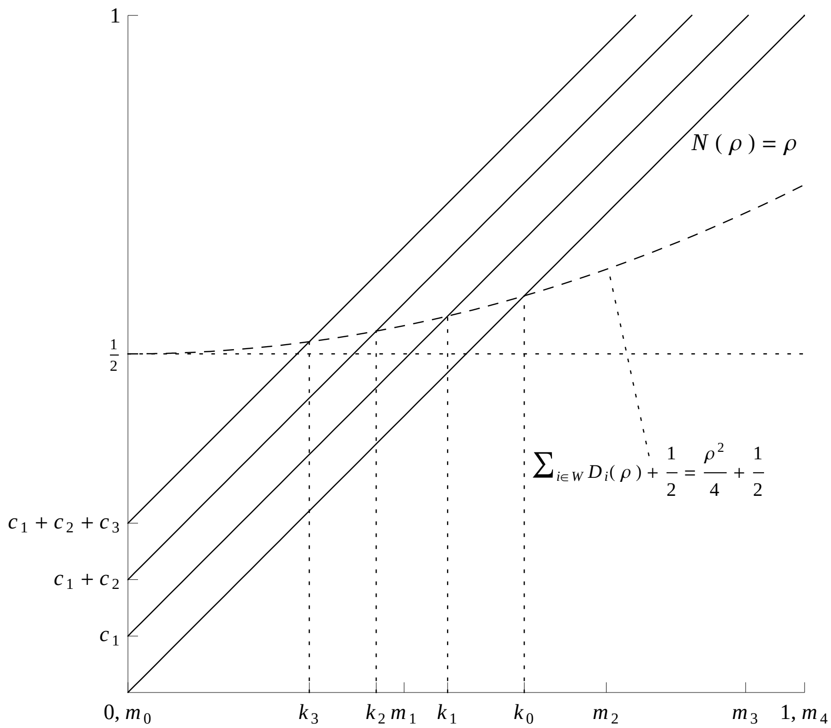

Figure 2: Above we see the how the cumulative distribution functions are related to the k_i. Specifically, we have the values for example 2 plotted above, note the Nash equilibrium candidate position $k_1 = \inf\{\max\{m_i, k_i\}\}$.

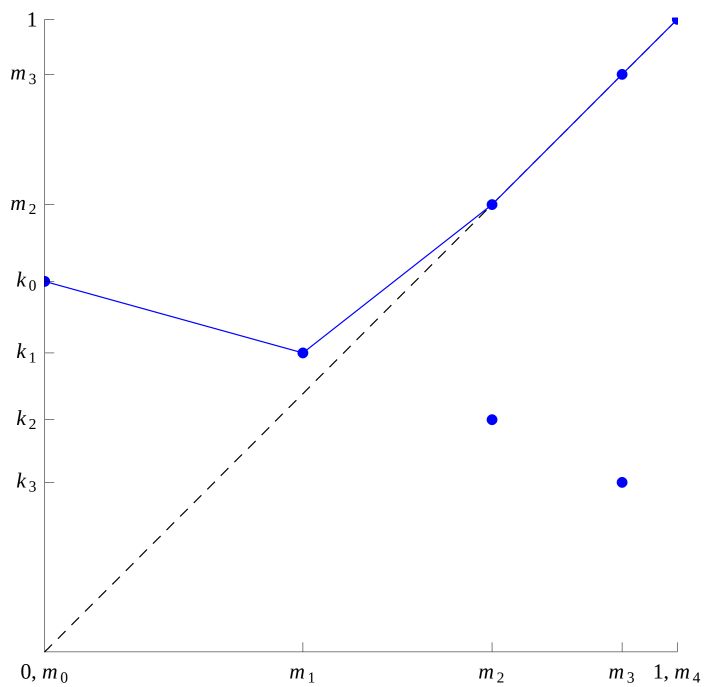

Figure 3: The more general specification of the Nash equilibrium candidate position from proposition 3.2 applied to example 2. Specifically the points in $\inf\{p \geq m_i : \int_0^p n(\rho) - \sum_{i \in \mathcal{W}} d_i(\rho)\, d\rho + \sum_{\{j \in \mathcal{W}: j \leq i\}} c_j \geq \frac{1}{2}\}$ (blue)are plotted and joined. We note how $k_1 = \inf\{\max\{m_i, k_i\}\}$ and the points (m_i, k_i) where $m_i > k_i$ (not joined).

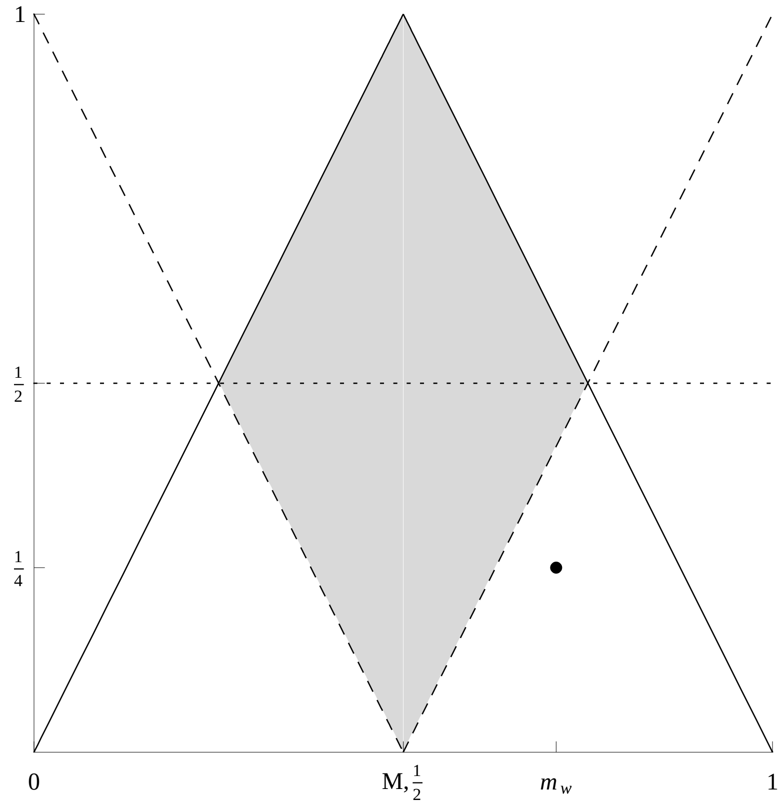

Figure 4: The shaded region is bounded by the functions specified in proposition 4.3 (dotted) and the constructible medians (solid), calculated for $n(\rho) = 1_{\{\rho \in [0,1]\}}$ as done in section 4. We note that m_w falls outside of the shaded region at $c_w = \frac{1}{4}$, implying that it could never be the case that a candidate assumes m_w in equilibrium, and further that the greatest skewing possible occurs at $c_w = \frac{1}{2}$.

References

Black, D. (1948). On the rationale of group decision-making [Electronic Version]. *The Journal of Political Economy, 5(1),* 23-34.

Hotelling, H. (1929). Stability in competition [Electronic Version]. *The Economic Journal, 39(153),* 41-57.

Osbourne, M. J., (2004). ***Introduction to game theory.*** New York, NY: Oxford University Press.