Solow Model & Sustainable Growth

Econ567, Spring 2013, Project I

Xiaolu Wang & Ying Guo

March 21, 2013

- I. Objective

- II. Model Development

  1) Solow model: saving rate

  2) Solow model: population growth

  3) Solow model: technological progress

- III. Main Conclusions and Policy Advice

Sustainable Growth Model

- Equation System

$$production\ \ function:? \qquad Y = AK^{a}L^{1-a}$$

$$transformed\ \ production:? \qquad y_t = A_t k_t^{a?}\ due\ to\ CRS \quad k = \frac{K}{L} \quad y = \frac{Y}{L}$$

$$production\ allocation:? \qquad y_{\mathrm{t}} = i_t + c_{\mathrm{t}}$$

$$investment\,equals\ \ savings:? \qquad i_t = s \times y_t$$

$$capital\ \ \ accumulation:? \qquad \Delta k_t = i_t - \delta k_t - nk_t$$

$$k_{\mathrm{t+1}} = k_t + \Delta k_t$$

$$population\ \ \ growth:? \qquad L = L_0$$

Symbol System

Exogenous Variables (set parameters)		
A	2	*technology parameter*
alpha	0.6	*capital exponent*
s	0.4	*savings rate per worker*
δ	0.6	*depreciation rate per worker*
initial k	6	*initial capital-labor ratio*

Endogenous Variables	
k	*capital per worker*
y	*output per worker*
c	*consumption per worker*
i	*investment per worker*
δk	*depreciation per worker*

Table 1: Equilibrium Condition: Δk=0 (population growth rate =0)

definition	k=K/L	y=Ak^a		c=(1-s)y	i=sy=sf(k)	δk=δ(K/L)	Δk=i-δk
Year	**k**	**y**	**%Δy**	**c**	**i**	**δk**	**Δk**
1	6	5.860312	——	3.5161873	2.3441248	3.6	-1.25588
2	4.744125	5.090069	-13.14%	3.0540413	2.0360276	2.846475	-0.81045
3	3.933678	4.54893	-10.63%	2.729358	1.819572	2.360207	-0.54063
4	3.393043	4.162782	-8.49%	2.4976689	1.6651126	2.035826	-0.37071
5	3.02233	3.883606	-6.71%	2.3301633	1.5534422	1.813398	-0.25996
10	2.271315	3.271873	-1.87%	1.9631238	1.3087492	1.362789	-0.05404
15	2.106565	3.127338	-0.49%	1.876403	1.2509353	1.263939	-0.013
20	2.066329	3.091361	-0.12%	1.8548165	1.2365443	1.239798	-0.00325
25	2.056224	3.082282	-0.03%	1.8493689	1.2329126	1.233735	-0.00082
30	2.053669	3.079982	-0.01%	1.8479893	1.2319929	1.232201	-0.00021
31	2.05346	3.079795	-0.01%	1.8478769	1.2319179	1.232076	-0.00016
32	2.053302	3.079652	0.00%	1.8477914	1.231861	1.231981	-0.00012
33	2.053182	3.079544	0.00%	1.8477265	1.2318177	1.231909	-9.14E-05
34	2.05309	3.079462	0.00%	1.8476771	1.2317848	1.231854	-6.95E-05

We start from a point with more capital per person than equilibrium. Capital depreciation exceeds capital accumulation, so that capital per person gradually decreased to equilibrium point. Since the 32 year, the economy enters a steady growth state with an equilibrium k* of 2.05309.

Graphical Description

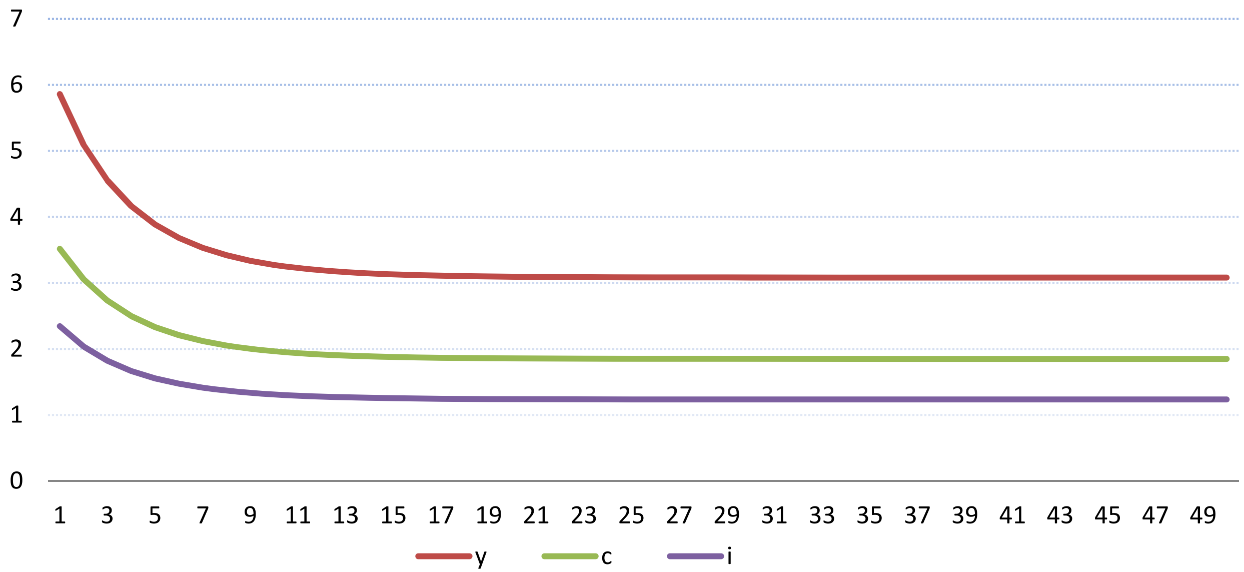

Figure.4 Income distribution over time

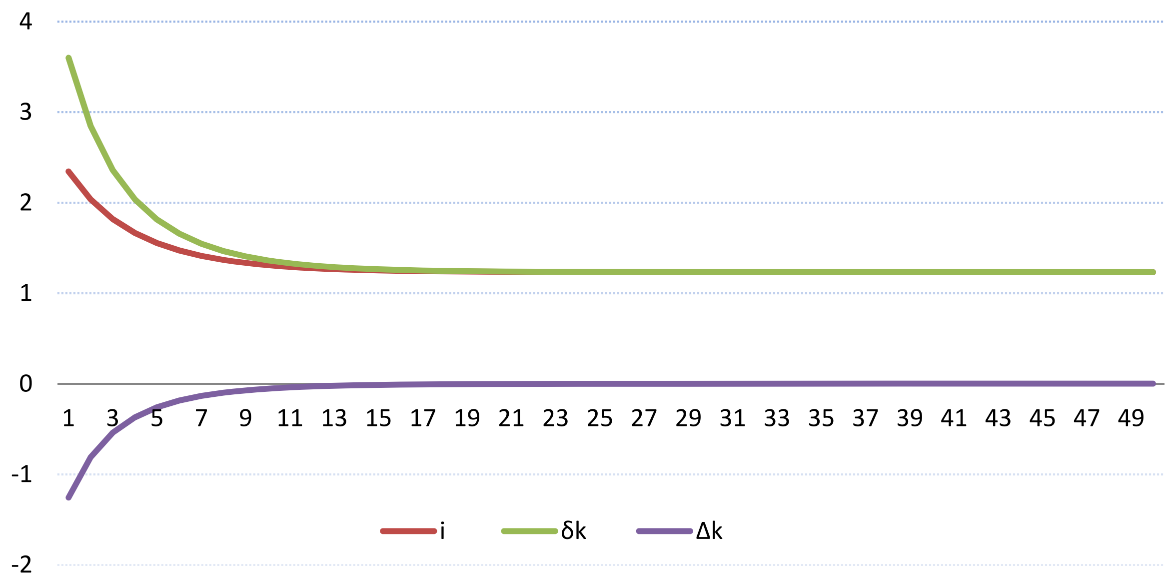

Figure.5 Capital and sustainable condition

Approach2: Algebra

$$The\ \ total\ \ change\ \ in\ \ capital-labor\ \ ratio\ \ is$$

$$\Delta k = sf(k) - \delta k$$

$$Our\ \ CRS\ \ Cobb-Douglas\ \ production\ \ function\ \ is$$

$$f(k) = Ak^{\alpha}$$

$$So\ \ we\ \ have: \Delta k = sAk^{\alpha} - \delta k$$

$$Equilibrium\ \ Condition: find\ \ the\ \ k^{*}$$

$$s.t.\ \ \Delta k^{*} = 0,\ \ i.e.,\ \ sAk^{*\alpha} - \delta k^{*} = 0$$

$$\therefore sAk^{*\alpha} = \delta k^{*}, rearrange: k^{*\alpha-1} = \frac{\delta}{sA}$$

$$\therefore k^{*} = \left(\frac{\delta}{sA}\right)^{\frac{1}{\alpha-1}}\ \ reduced\ \ form\ \ of\ \ the\ \ Solow\ \ Model$$

Solow Model: Initial Capital per person

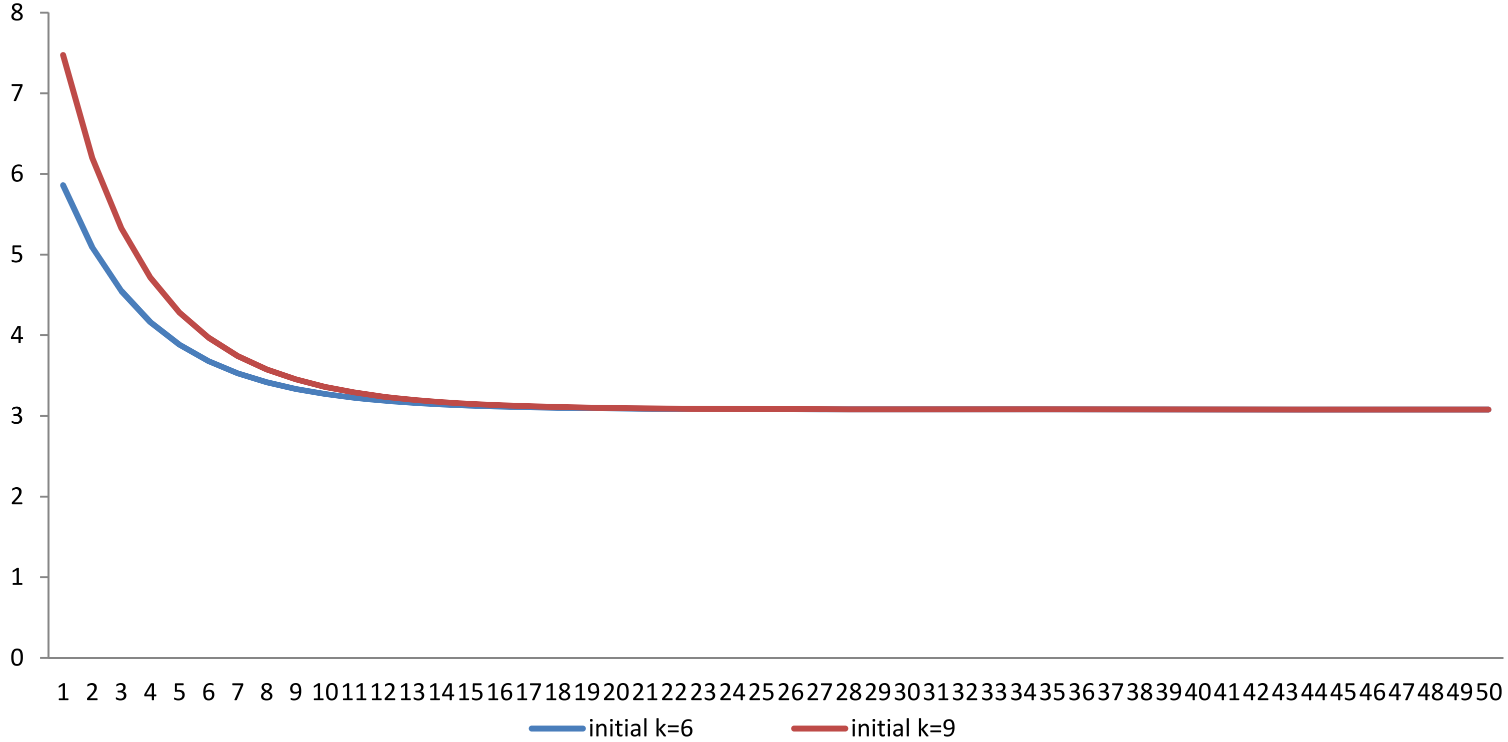

Figure. 6 Steady State of Output with Different Initial k

The equilibrium solution is not affected by initial value of capital per person since k*= (δ/sA)^(1/alpha-1). But it could influence the time to converge. Country with a higher initial capital per person may take longer to time to reach the equilibrium status.

Solow Model: Saving Rate

Economy		A				B		
saving rate		0.3				0.4		
Year	k	y	%Δy	Δk	k	y	%Δy	Δk
1	6	5.8603121	——	-1.841906	6	5.8603121	——	-1.255875
5	1.9789139	3.0122161	-11.08%	-0.283683	3.0223298	3.8836055	-6.71%	-0.259956
10	1.2006438	2.2319194	-3.32%	-0.05081	2.2713151	3.271873	-1.87%	-0.05404
15	1.0481639	2.0572521	-0.88%	-0.011723	2.1065646	3.1273383	-0.49%	-0.013003
20	1.0120433	2.0144173	-0.23%	-0.002901	2.0663292	3.0913609	-0.12%	-0.003253
25	1.0030428	2.0036492	-0.06%	-0.000731	2.0562244	3.0822815	-0.03%	-0.000822
30	1.0007708	2.0009249	-0.01%	-0.000185	2.0536686	3.0799822	-0.01%	-0.000208
31	1.0005858	2.0007029	-0.01%	-0.000141	2.0534603	3.0797948	-0.01%	-0.000158
32	1.0004452	2.0005342	-0.01%	-0.000107	2.0533021	3.0796524	0.00%	-0.00012
33	1.0003383	2.000406	-0.01%	-8.12E-05	2.0531818	3.0795442	0.00%	-9.14E-05
34	1.0002571	2.0003085	0.00%	-6.17E-05	2.0530904	3.0794619	0.00%	-6.95E-05
35	1.0001954	2.0002345	0.00%	-4.69E-05	2.0530209	3.0793994	0.00%	-5.28E-05

- We can see that the country with a higher saving rate converges to a stable economy more quickly and reaches equilibrium with higher output and higher capital per person.

Sustainable Growth Model-population & technology progress added

- Equation System

$$\textit{production function}:? \qquad Y = AK^{a}L^{1-a}$$

$$\textit{transformed production}:? \qquad y_t = A_t k_t^{a?} \textit{ due to CRS} \quad k = \frac{K}{L} \quad y = \frac{Y}{L}$$

$$\textit{Technology Progress}:? \qquad A_{t+1} = t \times A_t$$

$$\textit{production allocation}:? \qquad y_{\mathrm{t}} = i_t + c_{\mathrm{t}}$$

$$\textit{investment equals savings}:? \qquad i_t = s \times y_t$$

$$\textit{capital accumulation}:? \qquad \Delta k_t = i_t - \delta k_t - nk_t$$

$$k_{\mathrm{t}+1} = k_t + \Delta k_t$$

$$\textit{population growth}:? \qquad L_{t+1} = L_t \left(I + n \right)$$

Symbol System

Exogenous Variables (set parameters)		
alpha	0.6	*capital exponent*
s	0.4	*savings rate per worker*
δ	0.6	*depreciation rate per worker*
initial A	2	*initial technology parameter*
t	0	*technology progress rate*
n	0.01	*population growth rate*
initial k	6	*initial capital-labor ratio*

Endogenous Variables	
k	*capital per worker*
y	*output per worker*
c	*consumption per worker*
i	*investment per worker*
A	*technology parameter*
δk	*depreciation per worker*

Table 2: Equilibrium Condition: Δk=0 (population growth rate =0.01)

definition	k=K/L		y=Ak^alpha		c=(1-s)y	i=sy=sf(k)	δk=δ(K/L)		Δk=i-δk-nk
Year	**k**	**A**	**y**	**%Δy**	**c**	**i**	**δk**	**nk**	**Δk**
1	6	2	5.8603121	——	3.5161873	2.3441248	3.6	0.06	-1.315875
5	2.9217931	2	3.8055696	-6.98%	2.2833418	1.5222278	1.7530758	0.0292179	-0.260066
10	2.1778488	2	3.190411	-1.90%	1.9142466	1.2761644	1.3067093	0.0217785	-0.052323
15	2.0195701	2	3.0491961	-0.48%	1.8295176	1.2196784	1.211742	0.0201957	-0.012259
20	1.9819245	2	3.0149649	-0.12%	1.808979	1.205986	1.1891547	0.0198192	-0.002988
25	1.9727143	2	3.0065505	-0.03%	1.8039303	1.2026202	1.1836286	0.0197271	-0.000735
30	1.970445	2	3.004475	-0.01%	1.802684988	1.20179	1.182267	0.0197045	-0.000181
31	1.9702636	2	3.0043089	-0.01%	1.802585366	1.2017236	1.1821581	0.0197026	-0.000137
32	1.9701264	2	3.0041834	0.00%	1.802510054	1.2016734	1.1820758	0.0197013	-0.000104
33	1.9700227	2	3.0040885	0.00%	1.80245312	1.2016354	1.1820136	0.0197002	-7.84E-05
34	1.9699442	2	3.0040168	0.00%	1.802410079	1.2016067	1.1819665	0.0196994	-5.93E-05
35	1.969885	2	3.0039626	0.00%	1.80237754	1.201585	1.181931	0.0196988	-4.48E-05

- Assuming an annual population growth rate of 1%, the economy would enter a steady state since the 32 year.

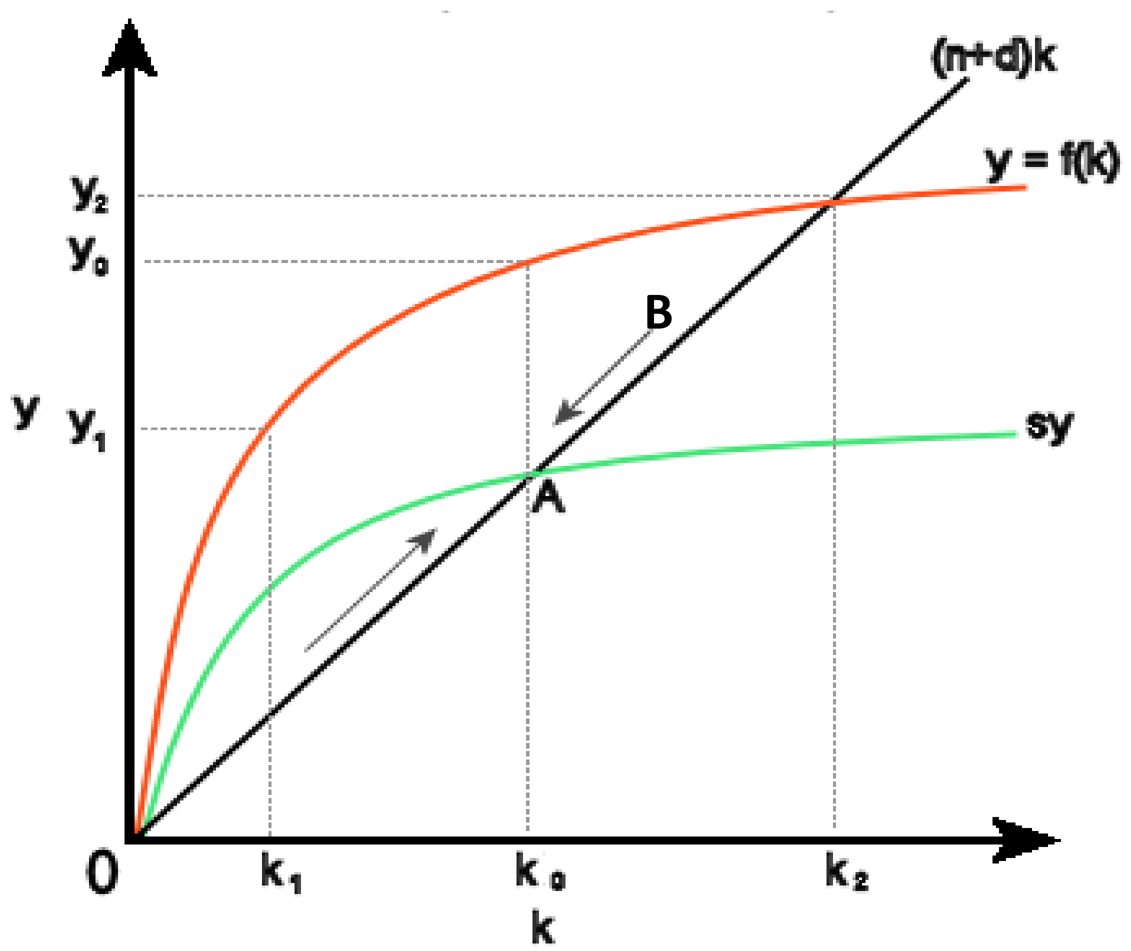

Figure. 7 Solow Growth Model

- We start from point B (Figure 6), where demand for capital, including capital for the newborn and refill for depreciated capital, exceeds savings.

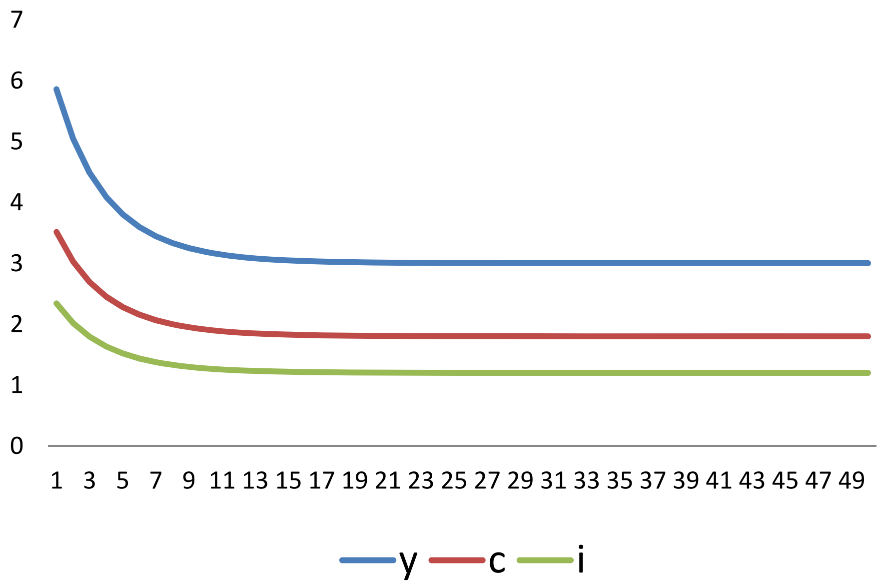

Figure.8 Income distribution over time with population growth

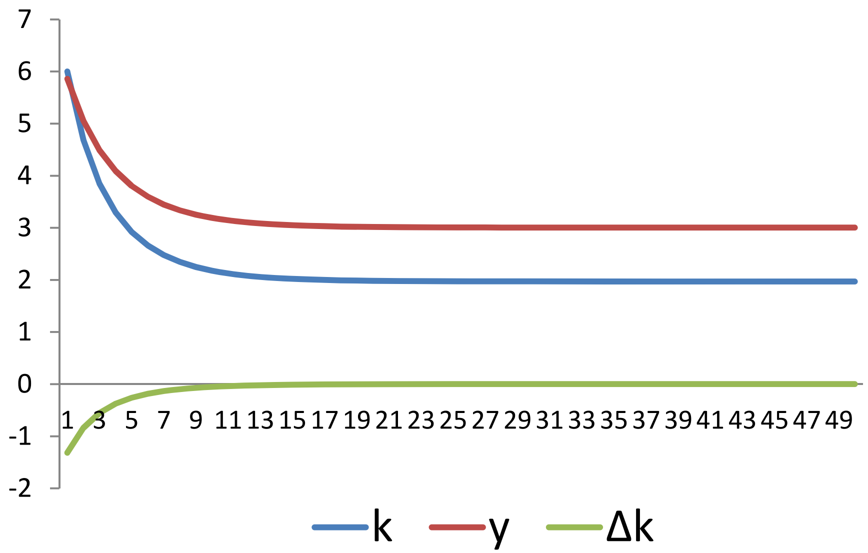

Figure.9 Capital and sustainable condition with population growth

- The capital available for everyone gradually declined and reached the equilibrium point around the 32nd year, where capital accumulation equals the demand for capital injection and the growth of output per capita is nearly zero.

Approach2: Algebra

$The\ \ total\ \ change\ \ in\ \ capital-labor\ \ ratio\ \ is$

$\Delta k = sf(k) - \delta k - nk$

$Our\ \ CRS\ \ Cobb-Douglas\ \ production\ \ function\ \ is$

$f(k) = Ak^{\alpha}$

$So\ \ we\ \ have: \Delta k = sAk^{\alpha} - \delta k - nk$

$Equilibrium\ \ Condition:\ find\ \ the\ \ k^{*}$

$s.t.\ \ \Delta k^{*} = 0,\ \ i.e.,\ \ sAk^{*\alpha} - \delta k^{*} - nk^{*} = 0$

$\therefore sAk^{*\alpha} = (\delta + n)k^{*}, rearrange: k^{*\alpha-1} = \dfrac{\delta + n}{sA}$

$\therefore k^{*} = \left(\dfrac{\delta + n}{sA}\right)^{\frac{1}{\alpha-1}}\ \ \ reduced\ \ form\ \ of\ \ the\ \ Solow\ \ Model$

Solow Model: Population Growth

Economy	A			B			C		
n	0			0.01			0.1		
Year	k	y	%Δy	k	y	%Δy	k	y	%Δy
1	6	5.8603121	——	6	5.86	——	6	5.86	——
28	2.0543032	3.0805533	-0.01%	1.971002726	3.00	-0.01%	1.3966304	2.44	-58.30%
29	2.0539426	3.0802288	-0.01%	1.970685124	3.00	-0.01%	1.3965389	2.44	0.00%
30	2.0536686	3.0799822	-0.01%	1.970445	3	-0.01%	1.396473	2.44	0.00%
31	2.0534603	3.0797948	-0.01%	1.9702636	3	-0.01%	1.3964256	2.44	0.00%
32	2.0533021	3.0796524	0.00%	1.9701264	3	0.00%	1.3963914	2.44	0.00%

- The country with slower population growth (B) reaches equilibrium at a similar speed as the country without population growth (A). This also applies to the situation where the economy converges to an equilibrium output higher than initial output.
- Economy with higher population growth rate will reach a equilibrium with lower capital and output per person. It seems that people's standard of living are pushed downwards in a country with fast population growth .

Solow Model: Technological Level

Economy		A			B			C	
A		2			4			4	
Year	**k**	**y**	**%Δy**	**k**	**y**	**%Δy**	**k**	**y**	**%Δy**
1	6	5.86	——	6	11.72	——	4	9.1895868	——
5	2.9217931	3.81	-6.98%	9.178498862	15.13	3.93%	8.1514315	14.08641	6.52%
10	2.1778488	3.19	-1.90%	10.62893243	16.52	0.92%	10.333508	16.240846	1.49%
15	2.0195701	3.05	-0.48%	11.01374179	16.87	0.23%	10.938061	16.804445	0.36%
20	1.9819245	3.01	-0.12%	11.11045244	16.96	0.06%	11.091594	16.945577	0.09%
25	1.9727143	3.01	-0.03%	11.13443859	16.98	0.01%	11.129771	16.980549	0.02%
29	1.9706851	3.00	-0.01%	11.13974057	16.99	0.00%	11.138215	16.988278	0.01%
30	1.970445	3.00	-0.01%	11.14036829	16.99	0.00%	11.139215	16.989193	0.01%
31	1.9702636	3.00	-0.01%	11.14084287	16.99	0.00%	11.139971	16.989885	0.00%
32	1.9701264	3.00	0.00%	11.14120166	16.99	0.00%	11.140542	16.990407	0.00%
33	1.9700227	3.00	0.00%	11.14147291	16.99	0.00%	11.140975	16.990803	0.00%
34	1.9699442	3.00	0.00%	11.14167798	16.99	0.00%	11.141301	16.991102	0.00%

Country with higher technology level converges to an equilibrium with higher capital and output per person. It also converges more quickly than the country with lower technology level.

Solow Model: Technological Progress

Economy			A					B		
t			0.01					0		
Year	**k**	**A**	**y**	**%Δy**	**Δk**	**k**	**A**	**y**	**%Δy**	**Δk**
1	6	2	5.86	——	-1.31587516	6	2	5.86	——	-1.315875
2	4.6841248	2.02	5.10	-12.94%	-0.81657258	4.6841248	2	5.05	-13.80%	-0.836778
5	3.0059918	2.081208	4.03	-5.28%	-0.2223856	2.9217931	2	3.81	-6.98%	-0.260066
10	2.4749671	2.1873705	3.77	0.43%	-0.00269261	2.1778488	2	3.19	-1.90%	-0.052323
15	2.5873257	2.2989484	4.07	2.00%	0.048402909	2.0195701	2	3.05	-0.48%	-0.012259
20	2.8735617	2.4162179	4.55	2.39%	0.067869331	1.9819245	2	3.01	-0.12%	-0.002988
25	3.2389102	2.5394693	5.14	2.49%	0.080355645	1.9727143	2	3.01	-0.03%	-0.000735
30	3.6637425	2.6690078	5.82	2.51%	0.091947823	1.970445	2	3.00	-0.01%	-0.000181
31	3.7556903	2.6956978	5.96	2.51%	0.094340214	1.9702636	2	3.00	-0.01%	-0.000137
32	3.8500305	2.7226548	6.11	2.51%	0.096775364	1.9701264	2	3.00	0.00%	-0.000104
33	3.9468059	2.7498814	6.27	2.52%	0.099258359	1.9700227	2	3.00	0.00%	-7.84E-05
34	4.0460642	2.7773802	6.42	2.52%	0.101793493	1.9699442	2	3.00	0.00%	-5.93E-05

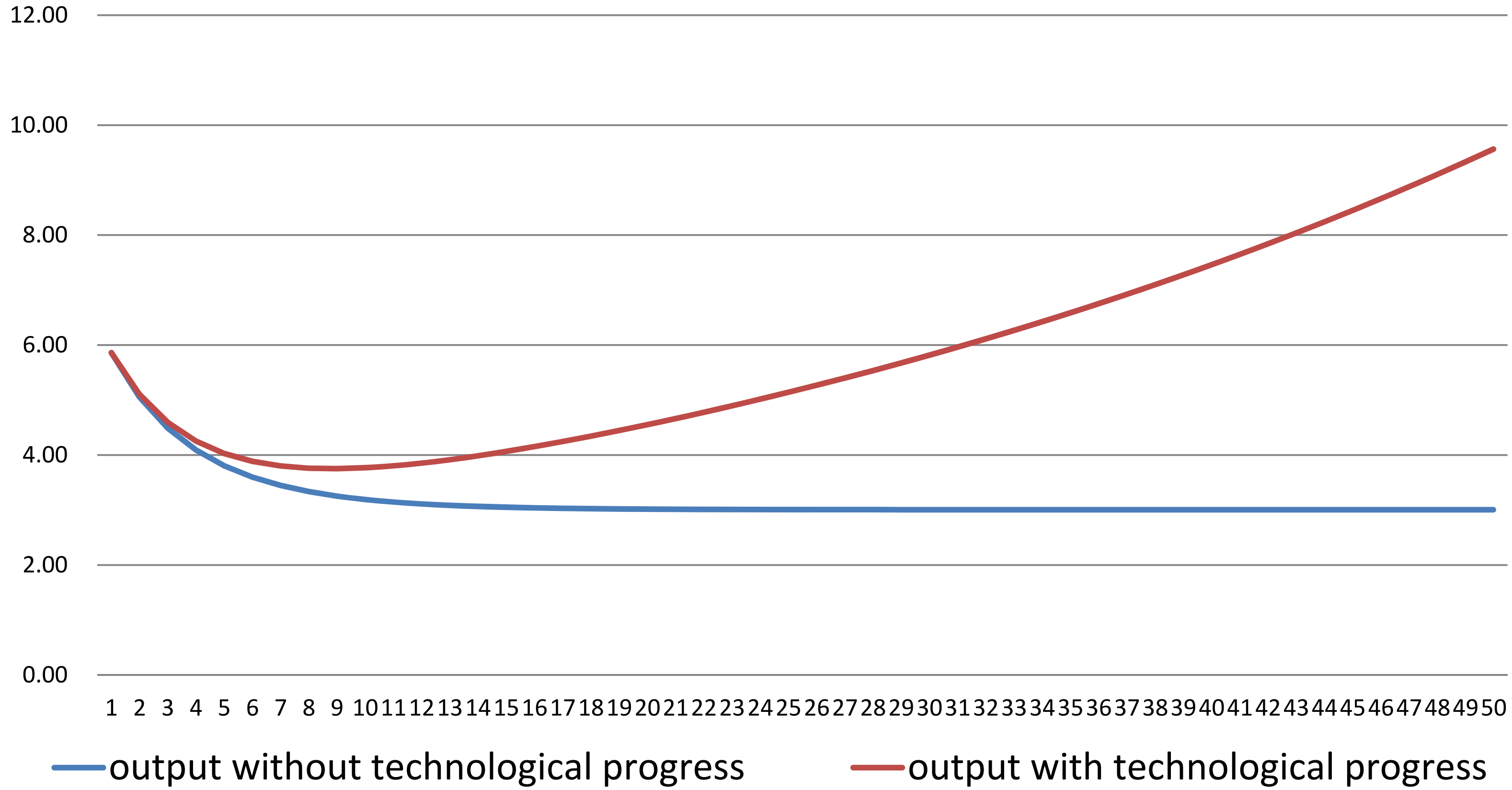

Figure.10 Output and Technological Progress

A country could not reach a stable growth status if there is also technological progress because the assumption of diminishing returns to capital may not hold. However, the country will enter a situation with constant growth rate of output per capita around the 33rd year.

III. Main Conclusions and Policy Advice

1. Country with a **higher saving rate** converges to a stable economy more quickly and reaches equilibrium with higher output per person. Government should encourage savings.
2. The country with **population growth** reaches equilibrium at a similar speed, but with lower output per person with population growth. Therefore, it is sensible for policy-makers to control population growth as to increase the welfare for everyone.
3. Country with **higher technology level** converges to equilibrium with higher capital and output per person. Advice: a developing country allocates some resources for technology development, even though at the cost of reduction of initial capital per person.
4. A country could **not reach a stable** growth status if there is also technological progress.
5. **Golden rule growth**. The idea that with depreciation and growth in the labor force, it is possible to get such a big capital stock that steady state consumption falls

- Thanks!